Final exams

Last Revision Date: April 3, 2024

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1. Past final exams

Final exam.

Name: ____ Group:______

Instructor's name: ____

Instructions

- Put your name in the blanks above (and also in all the paper provided).
- Solve **only two** of the three **long exercises** (2.5 points each), and answer **all** the **short questions** (0.5 points each)
- For each question, to receive full credit you must **show all work** (I have to be able to distinguish between a student who guesses the answers and one who understands the material). Explain your answers fully and clearly. If you add false statements to a correct argument, you will lose points.
- Calculators, books or notes of any form are **not** allowed.
- Total time: 2 hours

1.1. Final June 22/23

EXERCISE 1. A list containing only $\boldsymbol{u} = \begin{pmatrix}1\\1\\0\end{pmatrix}$ forms a basis of the subspace $\mathcal{W} = \{\boldsymbol{x} \in \mathbb{R}^n \mid \mathbf{A}\boldsymbol{x} = \mathbf{0}\}$ where $\underset{m\times n}{\mathbf{A}}$. You are asked to write

(a) (0.5^{pts}) The value of n (if it is possible). The rank of $\mathbf{A}$. The value of m (if it is possible) **(3 items!)**.
(b) (0.5^{pts}) The complete solution to $\mathbf{A}\boldsymbol{x} = \boldsymbol{c}$, where $\boldsymbol{c}$ is the first column of $\mathbf{A}$.
(c) (0.5^{pts}) A matrix $\mathbf{A}$ such that the list $[\boldsymbol{u};]$ forms a basis for $\mathcal{W}$.
(d) (0.5^{pts}) Some parametric equations of $\mathcal{V} = \{\boldsymbol{x} \in \mathbb{R}^n \mid \mathbf{BA}\boldsymbol{x} = \mathbf{0}\}$ where $\underset{m\times m}{\mathbf{B}}$ and $|\mathbf{B}| \neq 0$.
(e) (0.5^{pts}) Dimension and some Cartesian equations of the row space of $\mathbf{A}$.

EXERCISE 2. Consider $\underset{m\times n}{\mathbf{C}}$, with $m \geq n$, such that $\mathbf{C}^{\mathsf{T}}\mathbf{C}$ is invertible and consider $\boldsymbol{w}$, $\boldsymbol{u}$ and $\boldsymbol{z}$ such that:

$$\boldsymbol{w} = \boldsymbol{u} + \left(\mathbf{C}^{\mathsf{T}}\mathbf{C}\right)^{-1}(\mathbf{C}^{\mathsf{T}})\boldsymbol{z}.$$

(a) (0.5^{pts}) Show that the range of $\mathbf{C}$ is n.
Hint: Remember the relationship between rg($\mathbf{AB}$) and rg($\mathbf{B}$); or between the columns of $\mathbf{AB}$ and those of $\mathbf{A}$.
(b) (0.5^{pts}) If $\boldsymbol{w} = \boldsymbol{u}$, what is the relationship between $\boldsymbol{z}$ and the columns of $\mathbf{C}$?
(c) (0.5^{pts}) If $m = n$ and $\mathbf{C}$ is orthogonal, show that the length of $(\boldsymbol{w} - \boldsymbol{u})$ and $\boldsymbol{z}$ are equal.
(d) (0.5^{pts}) Is $\mathbf{C}^{\mathsf{T}}\mathbf{C}$ diagonalizable?
(e) (0.5^{pts}) Suppose that $\mathbf{C} = \begin{bmatrix}1&1&0\\0&0&0\\0&1&0\\0&0&1\end{bmatrix}$. Find the inverse of $\mathbf{C}^{\mathsf{T}}\mathbf{C}$.

EXERCISE 3. Let $\mathbf{A} = \mathbf{PD}(\mathbf{P}^{-1})$ where $\mathbf{P} = \begin{bmatrix}1&-1&1\\1&1&1\\1&0&-2\end{bmatrix}$ and $\mathbf{D} = \begin{bmatrix}1&0&0\\0&1&0\\0&0&2\end{bmatrix}$.

(a) (0.5^{pts}) Prove that columns of $\mathbf{P}$ are eigenvectors of $\mathbf{A}$ (explain your reasoning).
(b) (0.5^{pts}) If possible, find a matrix $\mathbf{C}$ (whose columns are not all perpendicular) and such that $\mathbf{A} = \mathbf{CD}(\mathbf{C}^{-1})$.
(c) (0.5^{pts}) If possible, find an orthogonal $\mathbf{Q}$ matrix such that $\mathbf{A} = \mathbf{QD}(\mathbf{Q}^{\mathsf{T}})$.
(d) (1^{pts}) Let $\mathbf{B} = \mathbf{A} - \mathbf{A}^{-1}$. Prove that $\mathbf{B}$ is the matrix of a quadratic form and find a polynomial expression for that quadratic form as a sum of squares.

Hint: *To answer these questions it is recommended not to explicitly find the matrix* $\mathbf{A}$.

SHORT QUESTIONS SET 1.

(a) Prove that the columns of $\underset{3\times 3}{\mathbf{B}}$ are an orthogonal basis of $\mathbb{R}^3$ if $\mathbf{B}^{\mathsf{T}}\mathbf{B} = \begin{bmatrix}3&0&0\\0&2&0\\0&0&6\end{bmatrix}$.
(b) The characteristic equation of $\underset{4\times 4}{\mathbf{A}}$ is $\lambda(\lambda-1)(\lambda^2-4) = 0$. Find all values of β for which the system $(\mathbf{A} - \beta\mathbf{I})\boldsymbol{x} = \mathbf{0}$ has a unique solution.
(c) Let $\mathbf{A} = \mathbf{BC}$ where $\mathbf{B} = \begin{bmatrix}1&2\\4&5\\2&7\end{bmatrix}$ and $\mathbf{C} = \begin{bmatrix}3&1&3\\1&2&1\end{bmatrix}$. Without finding $\mathbf{A}$, find a basis of $\mathcal{C}\left(\mathbf{A}\right)$ (justify your answer).

SHORT QUESTIONS SET 2. Let $\mathcal{H} = \mathcal{L}\left(\left[\begin{pmatrix}1\\1\\1\end{pmatrix}; \begin{pmatrix}1\\0\\1\end{pmatrix};\right]\right)$ and $\mathcal{G} = \mathcal{L}\left(\left[\begin{pmatrix}3\\2\\4\end{pmatrix}; \begin{pmatrix}0\\0\\1\end{pmatrix};\right]\right)$.

(a) Find some Cartesian equations for $\mathcal{H}$ and for $\mathcal{G}$.
(b) Find all vectors that are simultaneously in $\mathcal{H}$ and $\mathcal{G}$.

SHORT QUESTIONS SET 3. Let $\mathbf{P} = \begin{bmatrix}-1&-1&0&0\\-1&-2&0&0\\0&0&-1&-1\\0&0&-1&-a\end{bmatrix}$.

(a) Find the values of a for which P is invertible.

(b) Assuming P is invertible. For what values of a is $\boldsymbol{v} = \begin{pmatrix} 0 \\ 0 \\ -\frac{4}{3} \\ \frac{1}{3} \end{pmatrix}$ the third column of P^{-1}?

(c) Find the values of a for which the quadratic form $\boldsymbol{x}\mathsf{P}\boldsymbol{x}$ is negative definite.

Short questions set 4.

(a) Is it true the statement "If $\mathsf{BA} = \mathsf{A}$ then B is the identity matrix"? (Justify your answer).

(b) Let $\underset{3\times 2}{\mathsf{A}}$ and $\underset{2\times 4}{\mathsf{D}}$; and let C be full row rank. Could it be that $\mathsf{C} = \mathsf{AD}$? Give reasons for your answer.

1.2. Final May 22/23

EXERCISE 1. The solution set of $\underset{4\times 5}{\mathbf{A}}\ \boldsymbol{x} = \boldsymbol{b}$ is $\left\{ \boldsymbol{x} \in \mathbb{R}^5 \,\middle|\, \exists a, b \in \mathbb{R} : \boldsymbol{x} = \begin{pmatrix}1\\0\\0\\0\\0\end{pmatrix} + a\begin{pmatrix}1\\0\\1\\0\\0\end{pmatrix} + b\begin{pmatrix}1\\1\\0\\0\\0\end{pmatrix} \right\}$.

(a) (0.5^{pts}) What is the rank of $\mathbf{A}$? Is $\boldsymbol{b}$ equal to any of the columns of $\mathbf{A}$?
(b) (0.5^{pts}) If $\mathbf{E}$ is invertible and $\mathbf{B} = \mathbf{AE}$, is $\mathcal{C}(\mathbf{A})$ equal to $\mathcal{C}(\mathbf{B})$? (explain your answer)
(c) (0.5^{pts}) Is $\left[\begin{pmatrix}-1\\1\\1\\0\\0\end{pmatrix}; \begin{pmatrix}0\\0\\0\\1\\0\end{pmatrix}; \begin{pmatrix}0\\0\\0\\0\\1\end{pmatrix};\right]$ a basis of the subspace spanned by the rows of $\mathbf{A}$?
(d) (0.5^{pts}) Write Cartesian equations for the subspace formed by the solutions of $\mathbf{A}\boldsymbol{x} = \mathbf{0}$ (better think than calculate!).
(e) (0.5^{pts}) Does $\boldsymbol{v} = \begin{pmatrix}-1\\-2\\1\\0\\0\end{pmatrix}$, belong to the subspace that consists of all solutions of $\mathbf{A}\boldsymbol{x} = \mathbf{0}$? If so, write a basis for that subspace and write the coordinates of $\boldsymbol{v}$ in that basis.

EXERCISE 2. Let $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$, with $\mathbf{A} = \begin{bmatrix}1&0&1&a\\1&1&0&a\\0&1&-1&c\end{bmatrix}$, where a and c are parameters, and $\boldsymbol{b} = \begin{pmatrix}b_1\\b_2\\b_3\end{pmatrix}$.

(a) (0.5^{pts}) Discus for which values of a and c and which vectors $\boldsymbol{b}$ the system $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$ is solvable.
(b) (1^{pts}) Solve $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$ (with $\boldsymbol{b} = \begin{pmatrix}2, & 1, & -1,\end{pmatrix}$) for those values of a and c such that it is solvable.
(c) (0.5^{pts}) If possible, find the values of a and c for which $\mathsf{B} = \left[\begin{pmatrix}-1\\1\\1\\0\end{pmatrix}; \begin{pmatrix}0\\1\\1\\1\end{pmatrix};\right]$ is a basis of $\mathcal{N}(\mathbf{A})$.
(d) (0.5^{pts}) If possible, find the values a and c for which the $\mathcal{C}(\mathbf{A}^\intercal)$ equals the space spanned by the rows of $\mathbf{C} = \begin{bmatrix}1&-1&2&4\\1&-2&3&4\end{bmatrix}$.

EXERCISE 3. Let $\mathbf{A} = \begin{bmatrix}1&0&0&0\\2&2&0&0\\0&0&1&0\\2&b&0&1\end{bmatrix}$.

(a) (0.5^{pts}) Prove that $\mathbf{A}^{-1}$ exists.
(b) (0.5^{pts}) Find the values of b for which $\mathbf{A}^{-1}$ is diagonalizable (*note the inverse*).
(c) (0.5^{pts}) Using the values of b found in the previous section, find a basis of $\mathbb{R}^4$ consisting of eigenvectors of $\mathbf{A}$ (*note that* $\mathbf{A}$ *now appears instead of its inverse*).
(d) (1^{pts}) Let $\underset{4\times 4}{\mathbf{B}}$ with $|\mathbf{B}| = 2$. Find: $|2\mathbf{A}^\intercal\mathbf{B}|$, $|\mathbf{AB} - \mathbf{B}|$ and $\operatorname{tr}(\mathbf{BAB}^{-1})$

Hint: *You do not need to compute* $\mathbf{A}^{-1}$ *to answer any of these questions.*

SHORT QUESTIONS SET 1. *Determine whether the following statements are true or false, and explain your reasoning (to receive full credit you must explain your answers in a clear and concise way).*

(a) Let $\mathbf{B} = [\boldsymbol{v}_1; \boldsymbol{v}_2; \ldots \boldsymbol{v}_n;]$ be an eigenbasis of $\underset{n\times n}{\mathbf{A}}$ and let $\{\lambda_1, \lambda_2, \ldots \lambda_n\}$ be the corresponding eigenvalues. If the coordinates of $\boldsymbol{x}$ with respect to the basis $\mathbf{B}$ are $(a_1, a_2, \ldots a_n)$, that is, if $\boldsymbol{x} = \mathbf{B}\boldsymbol{a}$, then the coordinates of $\mathbf{A}\boldsymbol{x}$ with respect to $\mathbf{B}$ are $(\lambda_1 a_1, \lambda_2 a_2, \ldots \lambda_n a_n,)$
(b) Let $B = \{\boldsymbol{x}, \boldsymbol{y}\}$ be a basis for a subspace $\mathcal{S}$ of $\mathbb{R}^n$, and let $\underset{n\times n}{\mathbf{A}}$ be an invertible matrix. Then $B^* = \{\mathbf{A}\boldsymbol{x}, \mathbf{A}\boldsymbol{y}\}$ is a basis for $\mathcal{S}$.
(c) Any matrix $\mathbf{C}$, such that $\mathbf{C} = (\mathbf{A} + \mathbf{A}^\intercal)$ is diagonalizable.
(d) Let $f(\boldsymbol{x})$ be a quadratic form in $\mathbb{R}^n$ with associated matrix $\mathbf{A}$. If $\boldsymbol{v}$ is an eigenvector of $\mathbf{A}$ with eigenvalue $\lambda = 3$, and $f(\boldsymbol{v}) = 12$, then the length of $\boldsymbol{v}$ is 2.
(e) Let $\mathbf{A}$ be a 3×3 matrix with a determinant equal to 1, and let $\lambda_1 = 1$ and $\lambda_2 = 2$ be two of its eigenvalues. Then $\mathbf{A}^{-1}$ and $\mathbf{A}$ have the same trace.

Short questions set 2. Let $\mathbf{A}$ and $\mathbf{B}$ be 3×4 matrices. It is known that $|\mathbf{A}(\mathbf{B}^\intercal)| = 2$. Find:

(a) $\text{rg}\left(\mathbf{A}(\mathbf{B}^\intercal)\right)$, $\text{rg}\left(\mathbf{A}\right)$ and $\text{rg}\left(\mathbf{B}\right)$
(b) $|(\mathbf{B}^\intercal)\mathbf{B}|$, $|(\mathbf{B}^\intercal)\mathbf{A}|$ and $|\mathbf{B}(\mathbf{A}^\intercal)|$
(c) Solve the system of equations $\mathbf{A}(\mathbf{B}^\intercal)\boldsymbol{x} = \mathbf{0}$.
(d) Let $\mathbf{C} = \mathbf{A}(\mathbf{B}^\intercal)$, and let $\boldsymbol{c}_j$ denote the j-th column of $\mathbf{C}$. Find the determinant of the following matrix described by columns $\left[\boldsymbol{c}_2;\ (\boldsymbol{c}_2 - \boldsymbol{c}_1);\ (2\boldsymbol{c}_2 + \boldsymbol{c}_3);\ \right]$.

Short questions set 3. Let be the quadratic form $q\,(x,\ y,\ z) = by^2 + x^2 - 2xy + 2z^2$.

(a) Classify $q\,(x,\ y,\ z)$ based on the values of the parameter b.

1.3. Final July 21/22

EXERCISE 1. The following information is known about an $m \times n$ matrix $\mathbf{A}$:

$$\mathbf{A}\begin{pmatrix}1\\-2\\0\\1\end{pmatrix}=\begin{pmatrix}2\\4\end{pmatrix};\quad \mathbf{A}\begin{pmatrix}0\\2\\1\\3\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix};\quad \mathbf{A}\begin{pmatrix}2\\0\\0\\1\end{pmatrix}=\begin{pmatrix}5\\10\end{pmatrix};\quad \mathbf{A}\begin{pmatrix}3\\2\\0\\0\end{pmatrix}=\begin{pmatrix}1\\2\end{pmatrix}.$$

(a) (0.5^{pts}) Show that the vectors $\begin{pmatrix}1\\-2\\0\\1\end{pmatrix}, \begin{pmatrix}0\\2\\1\\3\end{pmatrix}, \begin{pmatrix}2\\0\\0\\1\end{pmatrix}$ y $\begin{pmatrix}3\\2\\0\\0\end{pmatrix}$ form a basis of $\mathbb{R}^4$.

(b) (0.5^{pts}) Give a matrix $\mathbf{C}$ and an invertible matrix $\mathbf{B}$ such that $\mathbf{A} = \mathbf{C}(\mathbf{B}^{-1})$. (You don't have to evaluate $\mathbf{B}^{-1}$ or find $\mathbf{A}$ explicitly. Just say what $\mathbf{B}$ and $\mathbf{C}$ are and use them to reason about $\mathbf{A}$ in the subsequent parts.)

(c) (1^{pts}) Find a basis for the null space of $\mathbf{A}^{\mathsf{T}}$.

(d) (0.5^{pts}) What are m, n, and the rank r of $\mathbf{A}$?

MIT 18.06 - Quiz 1, October 3, 2007

EXERCISE 2. Consider the matrix $\mathbf{A} = \begin{bmatrix}1&0&1&0\\0&1&0&1\\-1&0&1&0\\0&-1&0&1\end{bmatrix}$.

(a) (0.5^{pts}) Show that the columns of $\mathbf{A}$ are orthogonal to each other.

(b) (0.5^{pts}) Compute the determinant of $\mathbf{A}$.

(c) (0.5^{pts}) Prove that $\mathbf{A}^{-1} = \frac{1}{2}\mathbf{A}^{\mathsf{T}}$.

(d) (0.5^{pts}) Write down Cartesian equations of the subspace $\mathcal{S}$ of the linear combinations of the first two columns of $\mathbf{A}$

(e) (0.5^{pts}) Knowing that $\mathbf{A}^4 = -4\mathbf{I}$, which matrix is $\mathbf{A}^9$?

EXERCISE 3. A square matrix is said to be *stochastic* when all its elements are greater than or equal to zero and the sum of the components of each column is 1. Consider the stochastic matrix $\mathbf{A} = \begin{bmatrix}a & 1-b\\ 1-a & b\end{bmatrix}$, with $0 \le a \le 1$ and $0 \le b \le 1$.

(a) (0.5^{pts}) Show that $(1, \ 1,)$ is an eigenvector of $\mathbf{A}^{\mathsf{T}}$ (the transpose of $\mathbf{A}$) with associated eigenvalue $\lambda = 1$. Is $(1, \ 1,)$ an eigenvector of $\mathbf{A}$?

(b) (0.5^{pts}) Show that if $\boldsymbol{v}$ has components greater than or equal to zero and sum to 1, the components of $\mathbf{A}\boldsymbol{v}$ are also greater than or equal to zero and sum to 1.

(c) (0.5^{pts}) Show that all eigenvalues of $\mathbf{A}$ have absolute value less than or equal to 1. For what values of parameters a and b do both eigenvalues have absolute value equal to 1?
(*Hint:* use part a).

Cont. EXERCISE 3. *For the following two sections assume* $a = b$ *and* $0 < a < 1$.

(d) (0.5^{pts}) Find a basis $B = \{\boldsymbol{v}, \boldsymbol{w}\}$ of $\mathbb{R}^2$ that consists of eigenvectors of $\mathbf{A}$.

(e) (0.5^{pts}) Let B be the basis you have found in the previous section and let $\boldsymbol{x} = \alpha\boldsymbol{v} + \beta\boldsymbol{w}$; (with $\alpha, \beta \in \mathbb{R}$). Knowing that one of the eigenvalues is $\lambda_1 = 1$ and the other eigenvalue has absolute value less than 1, compute $\boldsymbol{z} = \lim_{k\to\infty} \mathbf{A}^k\boldsymbol{x}$. If the components of $\boldsymbol{z}$ sum to 1, what vector is $\boldsymbol{z}$?

SHORT QUESTIONS SET 1. Make up your own problem:

(a) Give an example of a matrix $\mathbf{A}$ and a vector $\boldsymbol{b}$ such that the solutions of $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$ form a line in $\mathbb{R}^3$, $\boldsymbol{b} \neq \mathbf{0}$, and all the entries of the matrix $\mathbf{A}$ are nonzero.

(b) Find all solutions to $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$.

SHORT QUESTIONS SET 2. Consider the matrix $\mathbf{A} = \begin{bmatrix}a&2&1\\2&a&1\\1&1&2\end{bmatrix}$, where a is a real number.

(a) For which values of the parameter a is $\mathbf{A}$ positive definite?

(b) For which values of the parameter a is the matrix $-\mathbf{A}$ positive definite?

(c) For which values of the parameter a is the matrix $\mathbf{A}$ singular?

MIT 18.06 - Quiz 3, May 07, 2007

SHORT QUESTIONS SET 3. Let $\mathbf{A} = \begin{bmatrix} 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

(a) What are the eigenvalues of $\mathbf{A}$?

(b) How many linearly independent eigenvectors does $\mathbf{A}$ have? Write a list of linearly independent eigenvectors of $\mathbf{A}$.

MIT 18.06 - Quiz 3, May 07, 2007

SHORT QUESTIONS SET 4.

(a) Given that: $\underset{3\times 3}{\mathbf{A}} \begin{bmatrix} 4 & 3 & 3 \\ -1 & -1 & -1 \\ -3 & 0 & 1 \end{bmatrix} = \mathbf{I}$, find the inverse of $\mathbf{A}^\mathsf{T}$.

MIT Course 18.06 Hour exam I, Fall 1996

SHORT QUESTIONS SET 5. *Determine whether the following statements are true or false, and explain your reasoning (to receive full credit you must explain your answers in a clear and concise way).*

(a) Let $\mathbf{A}$ be of order 2 by 3. If the solution set of $\mathbf{A}\boldsymbol{x} = \mathbf{0}$ is a line then the solution set of $(\mathbf{A}^\mathsf{T})\,\boldsymbol{y} = \mathbf{0}$ is a plane.

(b) If $\mathbf{A}$ is diagonalizable and the columns of $\mathbf{P}$ are a basis of eigenvector of $\mathbf{A}$, then the rows of $\mathbf{P}^{-1}$ are a basis of eigenvector of $\mathbf{A}^\mathsf{T}$.

1.4. Final May 21/22

EXERCISE 1. Consider $\mathbf{A} = \begin{bmatrix} 1 & 2 & 1 & 0 \\ 2 & 5 & 1 & 1 \\ 0 & 1 & 1 & 3 \end{bmatrix}$

(a) (0.5^{pts}) Which of the systems of linear equations $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$ or $\mathbf{A}^\mathsf{T}\boldsymbol{y} = \boldsymbol{c}$ can never have a unique solution (assuming there is a solution)?
(b) (0.5^{pts}) Which of the systems of linear equations $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$ or $\mathbf{A}^\mathsf{T}\boldsymbol{y} = \boldsymbol{c}$ might not have a solution? For that system of equations, give a right hand side vector ($\boldsymbol{b}$ or $\boldsymbol{c}$) for which a solution exists, and that has only *two nonzero entries* in the right-hand side.
(c) (0.5^{pts}) Write a basis of the subspace of all solutions of $\mathbf{A}\boldsymbol{x} = \mathbf{0}$.
(d) (0.5^{pts}) Consider the orthogonal complement of the previous subspace (i.e., the set of vectors perpendicular to the solutions of $\mathbf{A}\boldsymbol{x} = \mathbf{0}$). Write the Cartesian equations of that orthogonal complement.
(e) (0.5^{pts}) Write Cartesian equations for the subspace spanned by the solutions of $\mathbf{A}\boldsymbol{x} = \mathbf{0}$.

Based on MIT 18.06 Final Exam, Fall 2018

EXERCISE 2. The $m \times n$ matrix $\mathbf{A}$ has a factorization $\mathbf{A} = \mathbf{QR}$ where columns of $\mathbf{Q}$ are *orthonormal vectors* in $\mathbb{R}^m$ and $\mathbf{R} = \begin{bmatrix} 1 & -3 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$.

(a) (0.5^{pts}) Give as much true information as possible about m, n, and the rank of $\mathbf{A}$.
(b) (0.5^{pts}) Write the third column of $\mathbf{A}$ as a linear combination of the columns of $\mathbf{Q}$; explicitly write the coefficients of that linear combination.
(c) (0.5^{pts}) Compute the norm of the third column of $\mathbf{A}$.
(d) (0.5^{pts}) Are there columns of $\mathbf{A}$ orthogonal to each other? Which ones? Why? Why not? Do we have enough information to know it?
(e) (0.5^{pts}) If $\mathbf{A}$ is a square matrix, what is $|\det \mathbf{A}|$ (the absolute value of the determinant)?

Based on MIT 18.06 Final Exam, Fall 2018

EXERCISE 3.

(a) (1^{pts}) Find the inverse of $\mathbf{L} = \begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ 0 & a & 1 \end{bmatrix}$.

For all other items, consider $\mathbf{A} = \mathbf{LDL}^\mathsf{T}$ where $\mathbf{L}$ is the previous matrix and $\mathbf{D}$ is the diagonal matrix whose diagonal is $\left(d,\ d^2,\ d^3,\right)$. What are the conditions on a and d so that $\mathbf{A}$ is...

(b) (0.5^{pts}) invertible?
(c) (0.5^{pts}) symmetric?
(d) (0.5^{pts}) positive definite?

MIT 18.06 Final Exam, May 20, 2008

SHORT QUESTIONS SET 1. Consider $\mathbf{A} = \mathbf{XD}(\mathbf{X}^{-1})$, where $\mathbf{X} = \begin{bmatrix} 1 & 1 & -1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ and $\mathbf{D} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$;
and let be the matrix $\mathbf{M} = \mathbf{A}^4 - 2(\mathbf{A}^2) - 8\mathbf{I}$.

(a) Find the eigenvalues of $\mathbf{M}$.
(b) Solve $\mathbf{M}\boldsymbol{x} = \mathbf{0}$.

Hint: *You don't need to find* $\mathbf{M}$, *so better not calculate too much!*
Based on MIT 18.06 Final Exam, Fall 2018

SHORT QUESTIONS SET 2. *Determine whether the following statements are true or false, and explain your reasoning (to receive full credit you must explain your answers in a clear and concise way).*

(a) If $\mathbf{A}$ is invertible and symmetric, then $\mathbf{A}^{-1}$ is symmetric.
(b) If the columns of $\underset{m\times n}{\mathbf{Q}}$ (with $m > n$) are orthonormal, then $\mathbf{Q}(\mathbf{Q}^\mathsf{T})$ is invertible.
(c) If $\lambda = 0$ is an eigenvalue of $\mathbf{A}$ then the system of equations $\mathbf{A}\boldsymbol{x} = \mathbf{0}$ has infinitely many solutions.
(d) If $\mathbf{A} = \mathbf{A}^\mathsf{T}$ and $|\mathbf{A}| \neq 0$ then $\mathbf{A}^2$ is positive definite.
(e) If $\mathbf{A}$ y $\mathbf{B}$ are of order n, with the same trace ($\operatorname{tr}\left(\mathbf{A}\right) = \operatorname{tr}\left(\mathbf{B}\right)$) and the same determinant ($\det \mathbf{A} = \det \mathbf{B}$) then $\mathbf{A} = \mathbf{B}$.

(f) Any matrix with repeated eigenvalues is non-diagonalizable.
(g) The set of vectors containing only the null vector $\mathbf{0}$ is linearly independent.

Short questions set 3.

(a) Classify the quadratic form $f(x, y, z) = 2axz - x^2 - 4z^2$ for all possible values of the parameter a.

1.5. Final July 20/21

EXERCISE 1. Consider $\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 0 & 3 & 3 & 0 \\ 0 & 3 & 4 & 1 \\ 0 & 1 & 2 & 1 \end{bmatrix}$ and $\boldsymbol{b} = \begin{pmatrix} 1 \\ 2 \\ 0 \\ 3 \\ 3 \end{pmatrix}$.

(a) (1^{pts}) Solve the linear system $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$.
(b) (0.5^{pts}) Is the solution set a line in $\mathbb{R}^4$? Explain.
(c) (0.5^{pts}) Find a basis for $\mathcal{C}\left(\mathbf{A}^\mathsf{T}\right)$.
(d) (0.5^{pts}) Find a basis for the orthogonal complement of $\mathcal{C}\left(\mathbf{A}^\mathsf{T}\right)$.

EXERCISE 2.

(a) (1^{pts}) Find the eigenvalues of $\mathbf{A} = \begin{bmatrix} -1 & 2 & 4 \\ 0 & 0 & 5 \\ 0 & 0 & 1 \end{bmatrix}$, and an invertible matrix $\mathbf{S}$ whose columns are eigenvectors of $\mathbf{A}$.
(b) (0.5^{pts}) Explain why $\mathbf{A}^{1001} = \mathbf{A}$. Is $\mathbf{A}^{1000} = \mathbf{I}$?
(c) (0.5^{pts}) If we compute $\mathbf{A}^\mathsf{T}\mathbf{A}$ we get $\mathbf{A}^\mathsf{T}\mathbf{A} = \begin{bmatrix} 1 & -2 & -4 \\ -2 & 4 & 8 \\ -4 & 8 & 42 \end{bmatrix}$. How many eigenvalues of $\mathbf{A}^\mathsf{T}\mathbf{A}$ are positive? zero? negative? (Don't compute them but explain your answer).
(d) (0.5^{pts}) Does $\mathbf{A}^\mathsf{T}\mathbf{A}$ have the same eigenvectors as $\mathbf{A}$?

MIT Course 18.06 Quiz 3, Fall 2006

EXERCISE 3. Suppose the n by n matrix $\mathbf{A}$ has n orthonormal eigenvectors $\boldsymbol{q}_1, \dots, \boldsymbol{q}_n$ and n positive eigenvalues $\lambda_1, \dots, \lambda_n$. Thus $\mathbf{A}\boldsymbol{q}_j = \lambda_j \boldsymbol{q}_j$ and $\lambda_j > 0$, for $j = 1 : n$.

(a) (0.5^{pts}) What are the eigenvalues and eigenvectors of $\mathbf{A}^{-1}$? ***Prove that your answer is correct.***
(b) (1^{pts}) Any vector $\boldsymbol{b} \in \mathbb{R}^n$ is a linear combination of the eigenvectors:

$$\boldsymbol{b} = c_1\boldsymbol{q}_1 + \cdots + c_n\boldsymbol{q}_n.$$

Find a quick formula for c_1 using orthonormality of the $\boldsymbol{q}$'s.
(c) (1^{pts}) The solution to $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$ is also a linear combination of the eigenvectors:

$$\mathbf{A}^{-1}\boldsymbol{b} = d_1\boldsymbol{q}_1 + \cdots + d_n\boldsymbol{q}_n.$$

Find a quick formula for d_1. You can use the c's even if you didn't answer part (b).

MIT Course 18.06 Quiz 3, Fall 2006

SHORT QUESTIONS SET 1. Consider the 5 by 3 matrix $\mathbf{A}$ with orthonormal columns.

(a) Compute $\mathbf{A}^\mathsf{T}\mathbf{A}$
(b) What is the maximum possible value for the rank of $\mathbf{A}\mathbf{A}^\mathsf{T}$?
(c) Find $\det\left(\mathbf{A}\left(\mathbf{A}^\mathsf{T}\mathbf{A}\right)^{-1}\mathbf{A}^\mathsf{T}\right)$.

SHORT QUESTIONS SET 2. *Determine whether the following statements are true or false, and explain your reasoning (to receive full credit you must explain your answers in a clear and concise way).*

(a) If $\boldsymbol{u} = (1, \ 0, \ 0, \ 0,)$, $\boldsymbol{w} = (1, \ 1, \ 0, \ 0,)$ and $\mathcal{V} = \mathcal{L}(\boldsymbol{u}, \boldsymbol{w})$ is the bidimendisional subspace spanned by both vectors, then, the following are some Cartesian equations of $\mathcal{V}$:

$$\left\{2x + y = 0\,, \qquad \text{that is} \quad \left\{\boldsymbol{v} \in \mathbb{R}^4 \;\middle|\; \begin{bmatrix} 2 & 1 & 0 & 0 \end{bmatrix} \boldsymbol{v} = (0,)\right\}.\right.$$

(b) If $\mathbf{A}$ is invertible, then $\mathbf{A}^{-1} = \left(\mathbf{A}^\mathsf{T}\mathbf{A}\right)^{-1}\mathbf{A}^\mathsf{T}$.
(c) If $\boldsymbol{v}$ is an eigenvector of both matrix $\mathbf{A}$ and invertible matrix $\mathbf{B}$, then $\boldsymbol{v}$ is also an eigenvector of $\mathbf{A}\mathbf{B}^{-1}$.
(d) The set of vectors in $\mathbb{R}^3$ with integer (whole number) components is a subspace of $\mathbb{R}^3$.

SHORT QUESTIONS SET 3. Consider quadratic form $q(x, y) = 4xy$.

(a) Classify $q(x, y)$.
(b) Complete the square for the given quadratic form $q(x, y)$.

1.6. Final June 20/21

EXERCISE 1. Consider matrices $\mathbf{A} = \begin{bmatrix} 1 & -1 & 1 \\ -1 & a & -1 \\ 1 & -1 & 1 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 1 & a & 2 \\ 0 & 2 & 2 \\ 0 & 0 & 1 \end{bmatrix}$.

(a) (0.5^{pts}) Find the set of values for a such that $\mathbf{A}$ and $\mathbf{B}$ are diagonalizable.
(b) (0.5^{pts}) Find: $\operatorname{tr}\left(\mathbf{B}\mathbf{A}\mathbf{B}^{-1}\right)$ and $|\mathbf{A}\mathbf{B}^2|$ (where $\operatorname{tr}(\mathbf{M})$ is the trace of $\mathbf{M}$).
(c) (0.5^{pts}) Classify the quadratic form $\boldsymbol{x}\mathbf{A}\boldsymbol{x}$ in terms of a.
(d) (0.5^{pts}) Let $a = 1$. Find a diagonal matrix $\mathbf{D}$ and a matrix $\mathbf{S}$ such that $\mathbf{B}^3 = \mathbf{S}\mathbf{D}\mathbf{S}^{-1}$.
(*Note the exponent* 3)
(e) (0.5^{pts}) Let $a = 1$. Find an orthonormal basis for $\mathbb{R}^3$ formed by eigenvectors of $\mathbf{B}$, or explain why it is not possible to find such a basis.

EXERCISE 2. Let $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$; where $\mathbf{A} = \begin{bmatrix} 1 & a & 2 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{bmatrix}$ and $\boldsymbol{b} = \begin{pmatrix} c \\ 0 \\ 0 \end{pmatrix}$.

(a) (0.5^{pts}) For which values of a and c is $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$ solvable? For which values of a and c is the solution to $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$ unique?
(b) (1^{pts}) Solve the system for those values of parameters a and c that make the system solvable.
(c) (0.5^{pts}) Is it possible to express the set of solutions so that the free variables are...

A) x_2 and x_3?

B) x_4 and x_5?

(d) (0.5^{pts}) Find $|\mathbf{A}^{\intercal}\mathbf{A}|$.

EXERCISE 3. Suppose you have a 3×3 matrix $\mathbf{A}$ satisfying $\mathbf{A} = \mathbf{B}^{-1}\mathbf{U}\mathbf{L}$ where

$$\mathbf{B} = \begin{bmatrix} -7 & 2 & 9 \\ 13 & -1 & 1 \\ -2 & 0 & -17 \end{bmatrix}; \qquad \mathbf{U} = \begin{bmatrix} 1 & -3 & 7 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}; \qquad \mathbf{L} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 5 & -12 & 1 \end{bmatrix}.$$

Important: Do not compute $\mathbf{A}^{-1}$!

(a) (0.5^{pts}) The second column $\boldsymbol{c}$ of the matrix $\mathbf{A}^{-1}$ satisfies $\mathbf{A}\boldsymbol{c} = \boldsymbol{b}$ for what right-hand side $\boldsymbol{b}$?
(b) (1^{pts}) The second column $\boldsymbol{c}$ of the matrix $\mathbf{A}^{-1}$ also satisfies $\mathbf{U}\mathbf{L}\boldsymbol{c} = \boldsymbol{d}$ for what right-hand side $\boldsymbol{d}$?
(c) (1^{pts}) Compute the second column $\boldsymbol{c}$ of the matrix $\mathbf{A}^{-1}$ solving $\mathbf{U}\mathbf{L}\boldsymbol{c} = \boldsymbol{d}$ (or any equivalent system).

MIT 18.06 - Quiz 1, Fall 2017

SHORT QUESTIONS SET 1. Let $\mathbf{A}$ be a matrix of order 2×5 such that $|\mathbf{A}\mathbf{A}^{\intercal}| = 3$. (*explain your answers*)

(a) Is there an $\boldsymbol{x} \neq \mathbf{0}$ such that $(\mathbf{A}^{\intercal})\boldsymbol{x} = \mathbf{0}$? Is there an $\boldsymbol{x} \neq \mathbf{0}$ such that $\mathbf{A}\boldsymbol{x} = \mathbf{0}$?
(b) Find $|2(\mathbf{A}\mathbf{A}^{\intercal})^{-1}|$.
(c) Find $\det\left[\boldsymbol{c}_2;\ (3\boldsymbol{c}_1 + 2\boldsymbol{c}_2)\right]$ where $\boldsymbol{c}_j$ is the j-th column of $\mathbf{A}\mathbf{A}^{\intercal}$.
(d) Find the dimension of $\mathcal{S} = \left\{\boldsymbol{x} \in \mathbb{R}^2 \mid \mathbf{A}(\mathbf{A}^{\intercal})\boldsymbol{x} = \mathbf{0}\right\}$.
(e) Find the dimension of $\mathcal{W} = \left\{\boldsymbol{x} \in \mathbb{R}^5 \mid (\mathbf{A}^{\intercal})\mathbf{A}\boldsymbol{x} = \mathbf{0}\right\}$.
(f) Find the dimension of $\mathcal{Z} = \mathcal{L}\left(\left[\boldsymbol{f}_1;\ \dots \boldsymbol{f}_5;\right]\right)$, where $\boldsymbol{f}_i$ is the i-th row of $\mathbf{A}^{\intercal}\mathbf{A}$.
(g) Is $\lambda = 0$ an eigenvalue of $\mathbf{A}^{\intercal}\mathbf{A}$? If so, what is its geometric multiplicity? and its algebraic multiplicity? Otherwise, why $\lambda = 0$ is not an eigenvalue?
(h) Prove that $\boldsymbol{x}\mathbf{A}\left(\mathbf{A}^{\intercal}\right)\boldsymbol{x}$ is positive definite.

SHORT QUESTIONS SET 2. Consider the plane P containing vector $\boldsymbol{q} = \begin{pmatrix}1, & 0, & 1,\end{pmatrix}$ and parallel to vectors $\boldsymbol{u} = \begin{pmatrix}1, & -1, & 1,\end{pmatrix}$ and $\boldsymbol{v} = \begin{pmatrix}2, & 1, & -1,\end{pmatrix}$.

(a) Write Cartesian equations for P.
(b) Which of the following vectors belong to P? $\boldsymbol{a} = \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix}$; $\boldsymbol{b} = \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$; $\boldsymbol{c} = \begin{pmatrix} 5 \\ 0 \\ 1 \end{pmatrix}$ y $\boldsymbol{d} = \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}$.

1.7. Final June 18/19

EXERCISE 1. Consider $\mathbf{A}$ such that $\mathbf{AB} = \mathbf{C}$, where $\mathbf{B} = \begin{bmatrix} 1 & -1 & 1 & 4 & 0 \\ 0 & 6 & 0 & 0 & 6 \\ 2 & 4 & 0 & 6 & 6 \end{bmatrix}$ and $\mathbf{C} = \begin{bmatrix} 0 & 0 & 2 & 2 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 3 & 1 \end{bmatrix}$.

(a) (0.5^{pts}) Prove that the third column of $\mathbf{C}$ is equal to one of the columns of $\mathbf{A}$.
(b) (0.5^{pts}) Is $\mathbf{A}$ invertible? Please explain.
(c) (0.5^{pts}) Solve $\mathbf{A}\boldsymbol{x} = \mathbf{C}_{|5}$ (where $(\mathbf{C}_5)_{|i}$ s the fifth column of $\mathbf{C}$).
(d) (0.5^{pts}) Are rows of $\mathbf{C}$ a basis for $\mathcal{C}\left(\mathbf{B}^{\intercal}\right)$? (the subspace spanned by rows of $\mathbf{B}$)
(e) (0.5^{pts}) Is $\mathbf{A}$ diagonalizable? (please explain). If it is so, find a basis for $\mathbb{R}^3$ formed by eigenvectors of $\mathbf{A}$ (and writedown the corresponding eigenvalues). (*hint:* look at the last columns of $\mathbf{B}$ and $\mathbf{AB} = \mathbf{C}$).

Remark: *none of the above questions require to find* $\mathbf{A}^{-1}$ *or* $\mathbf{A}$*; we only need to note that* $\mathbf{AB} = \mathbf{C}$ *and inspect columns of* $\mathbf{B}$ *and* $\mathbf{C}$.

EXERCISE 2. Consider $\mathbf{A}$ of order 3, such that $\mathbf{A}\boldsymbol{v}_1 = \boldsymbol{v}_1$, $\mathbf{A}\boldsymbol{v}_2 = \mathbf{0}$, and $\mathbf{A}\boldsymbol{v}_3 = \boldsymbol{v}_3$; where

$$\boldsymbol{v}_1 = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, \qquad \boldsymbol{v}_2 = \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}, \qquad \boldsymbol{v}_3 = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}.$$

Without explicitly computing $\mathbf{A}$, answer the following questions:

(a) (0.5^{pts}) Find the solution set for $\mathbf{A}\boldsymbol{x} = \mathbf{0}$.
(b) (0.5^{pts}) ¿Is $\mathbf{A}$ symmetric? ¿Is $\mathbf{A}$ diagonalizable? (explain your answer).
(c) (0.5^{pts}) Find an orthonormal basis for the eigenspace corresponding to the eigenvalue that has algebraic multiplicity equal to two.
(d) (0.5^{pts}) Compute $\mathbf{A}^k\boldsymbol{x}$ for all $k \neq 0$ when $\boldsymbol{x} = (2\boldsymbol{v}_1 - \boldsymbol{v}_3)$.
(e) (0.5^{pts}) Find $\boldsymbol{x}\mathbf{A}\boldsymbol{x}$, when $\boldsymbol{x} = (2\boldsymbol{v}_1 - \boldsymbol{v}_3)$.

EXERCISE 3. Consider the full column rank matrix $\underset{m\times n}{\mathbf{X}}$ (with $n < m$), and let $\mathbf{P}$ be the projection matrix onto $\mathcal{C}\left(\mathbf{X}\right)$.

Are the followings statements true or false? Prove the statement when it is true, or justify why it is false.

(a) (0.5^{pts}) $\mathbf{P}$ is orthogonal.
(b) (0.5^{pts}) $\mathbf{P}$ is symmetric.
(c) (0.5^{pts}) $\mathbf{P}$ is idempotent.
(d) (1^{pts}) $\left(\boldsymbol{v} - \mathbf{P}\boldsymbol{v}\right)$ is orthogonal to $\mathbf{P}\boldsymbol{v}$ for any $\boldsymbol{v} \in \mathbb{R}^n$.

Basado en una propuesta de Manuel Morán

SHORT QUESTIONS SET 1.

(a) Consider two orthogonal matrices $\mathbf{A}$ and $\mathbf{B}$ such that $\mathbf{C} = \mathbf{AB}$ is a symmetric matrix. Prove $\mathbf{C}$ is *unipotent* (that is, prove that $\mathbf{C}^2 = \mathbf{I}$)
(b) ¿Is the system of vectors $(1, -2, 0, 1)$ and $(1, 0, 2, 1)$ a basis for the solution set of $\begin{cases} 2x_1 + x_2 - x_3 & = 0 \\ x_1 \qquad\qquad - x_4 & = 0 \end{cases}$? Please justify.
(c) Find the cartesian (or implicit) equations for the spam of $\{(1, \; 2, \; 0,), (2, \; 1, \; 1,), (1, \; -1, \; 1,)\}$.
(d) Find a parametric representation of the line that goes through $(1, \; 2, \; 0,)$ and $(2, \; 1, \; 1,)$.

SHORT QUESTIONS SET 2. Consider $\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & -1 & 1 \\ -2 & 0 & 0 & 0 \end{bmatrix}$, and $\boldsymbol{b} = \begin{pmatrix} 1 \\ 3 \\ 2 \\ -1 \end{pmatrix}$.

(a) Is $\mathbf{A}$ invertible? Please justify. If it is so, find the component $(3, 2)$ of $\mathbf{A}^{-1}$.
(b) Find the third componnet x_3 of the solution to $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$.

SHORT QUESTIONS SET 3. Consider the quadratic form $q(x, y, z) = ax^2 + ay^2 + 2xz + az^2$ where a is a parameter.

(a) Find the corresponding matrix $\mathbf{A}$ such that $q(\boldsymbol{x}) = \boldsymbol{x}\mathbf{A}\boldsymbol{x}$. For what values of parameter a is $\mathbf{A}$ diagonalizable?

(b) Are $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ eigenvectors of $\mathbf{A}$?

(c) Classify the quadratic form depending on the parameter a.

(d) For $a = 2$ write down $q(x, y, z)$ as a sum of squares.

1.8. Final May 18/19

EXERCISE 1. Consider $\mathcal{S} = \mathcal{L}\{\boldsymbol{u}, \boldsymbol{v}\} \subset \mathbb{R}^3$, the spam of $\boldsymbol{u} = \begin{pmatrix}1\\1\\0\end{pmatrix}$ and $\boldsymbol{v} = \begin{pmatrix}0\\1\\1\end{pmatrix}$, and consider $\mathcal{S}^\perp \subset \mathbb{R}^3$ the orthogonal complement of $\mathcal{S}$:

$$\mathcal{S}^\perp = \left\{\boldsymbol{x} \in \mathbb{R}^3 \,\middle|\, \boldsymbol{x} \cdot \boldsymbol{w} = 0 \text{ for all } \boldsymbol{w} \in \mathcal{S}\right\} = \left\{\boldsymbol{x} \in \mathbb{R}^3 \,\middle|\, \mathbf{S}^\intercal \boldsymbol{x} = \mathbf{0} \text{ where } \mathbf{S} = \left[\boldsymbol{u}\boldsymbol{v}\right]\right\}.$$

(a) (0.5^{pts}) Find an orthonormal basis for $\mathcal{S}$.
(b) (0.5^{pts}) Find a set of parametric equations for $\mathcal{S}^\perp$.

For the remaining of this exercise, $\mathbf{P}$ is the orthogonal projection matrix onto $\mathcal{S}$

(c) ($0.5 + 0.5^{\text{pts}}$) Prove that non-zero vectors in $\mathcal{S}$ and non-zero vectors in $\mathcal{S}^\perp$ are eigenvectors of $\mathbf{P}$.
(d) (0.5^{pts}) Find an orthogonal matrix $\mathbf{Q}$ and a diagonal matrix $\mathbf{D}$ such that $\mathbf{P} = \mathbf{Q}\mathbf{D}\mathbf{Q}^\intercal$

Note: *in order to answer the two last questions, you don't need to find* $\mathbf{P}$*. Since* $\mathbf{P}$ *is the orthogonal projection matrix onto* $\mathcal{S}$*, it is enough to understand what is* $\mathbf{P}\boldsymbol{w}$ *in two cases: when* $\boldsymbol{w} \in \mathcal{S}$ *and when* $\boldsymbol{w} \in \mathcal{S}^\perp$*.*

EXERCISE 2. Consider the system $\begin{cases} x_1 - x_2 + x_3 + 2x_4 = b \\ x_1 \qquad + x_3 + 2x_4 = 0 \\ ax_1 + x_2 + x_3 + 2x_4 = 0 \end{cases}$

(a) (0.5^{pts}) For what values of a and b the system is unsolvable, solvable with a unique solution or solvable with infinite solutions?

For the remaining of this exercise $a = 1$ and $b = 0$.

(b) (0.5^{pts}) Solve the system.
(c) ($0.5 + 0.5^{\text{pts}}$) Find a *basis* for the set of solutions and find the coordinates[1] of $(1, 0, 1, -1)$ respect to that basis.
(d) (0.5^{pts}) Is it posible to take x_1 y x_4 as free variables? Justify your answer.

EXERCISE 3. Consider the matrix $\mathbf{A} = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix}$ and the two systems of equations (S1) and (S2):

$$\text{(S1):} \quad \mathbf{A}\boldsymbol{x} - \alpha \begin{pmatrix}1\\1\\1\\1\end{pmatrix} = \boldsymbol{b} \qquad\qquad \text{(S2):} \quad \left(\mathbf{A} - \alpha\mathbf{I}\right)\boldsymbol{x} = \mathbf{0}$$

where $\boldsymbol{x}$ is a vector of unknowns (in both systems), $\mathbf{I}$ is the identity matrix, $\boldsymbol{b} \in \mathbb{R}^4$ and $\alpha \in \mathbb{R}$.

(a) (0.5^{pts}) For what scalars α and vectors $\boldsymbol{b}$ the set of solutions in (S1) is a subspace of $\mathbb{R}^4$?
(b) (0.5^{pts}) Find the rank of $\mathbf{A}$.
(c) (0.5^{pts}) Decide if $\boldsymbol{v} = \begin{pmatrix}1\\1\\1\\1\end{pmatrix}$ and $\boldsymbol{w} = \begin{pmatrix}1\\-1\\-1\\1\end{pmatrix}$ are solutions to (S2) for some values of α.
(d) (0.5^{pts}) Let $\boldsymbol{u} = p\boldsymbol{v} + q\boldsymbol{w}$, where $\boldsymbol{v}$ and $\boldsymbol{w}$ are the vectors in (c) and p and q are constants. Prove that, independently of p and q; vector $\mathbf{A}^3\boldsymbol{u}$ belongs to the line spanned by $\boldsymbol{v}$.
(e) (0.5^{pts}) Find the polynomial expression of the quadratic form corresponding to $\mathbf{A}$ and classify.

SHORT QUESTIONS SET 1.

(a) Find the orthogonal projection matrix onto the line spanned by $\begin{pmatrix}1, & 1, & -1, & 1\end{pmatrix}$.
(b) Write a set of parametric equations for the line that goes through $\begin{pmatrix}0, & 0, & 1\end{pmatrix}$ and is parallel to the span of $\begin{pmatrix}1, & 2, & 4\end{pmatrix}$.

[1] coordinates are the coeficients of the linear combination that equals $(1, 0, 1, -1)$

(c) Let $\mathbf{A}$ be a 3 by 3 matrix. Find two elementary matrices and explain the right order in which the three matrices are multiplied in order to get the following transformations: first we want to substract the first *row* from the second **row**, and then we want to multiply the second **column** by 4. Do we get a different result if we first multiply the second **column**, and then we substract the first **row** from the second one? Explain.

(d) Check if 2 is an eigenvalue for $\begin{bmatrix} 3 & 0 & 1 & -1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 2 & 0 \\ 2 & 0 & 2 & 0 \end{bmatrix}$. If so, find the corresponding eigenspace.

SHORT QUESTIONS SET 2. *Determine whether the following statements are true or false, and explain your reasoning (to receive full credit you must explain your answers in a clear and concise way).*

(a) if $\mathbf{A}$ and $\mathbf{B}$ are symmetric matrices and $\mathbf{B}$ is invertible, then $\mathbf{A}\left(\mathbf{B}^{-1}\right)$ is also symmetric.
(b) Let $\mathbf{A}$ be a matrix of order n. If $\mathbf{A}\boldsymbol{x} = \mathbf{0}$ has infinite solutions, then $\mathbf{A}$ is not orthogonal.
(c) If $\underset{n\times n}{\mathbf{P}}$ is symmetric and idempotent, then $\left(\mathbf{I} - \mathbf{P}\right)$ is also symetric and idempotent.
(d) If $[\boldsymbol{v},\ \boldsymbol{w},\ \boldsymbol{u}]$ is a basis of subspace $\mathcal{S}$, then so it is the system $[2\boldsymbol{v},\ (\boldsymbol{w}+\boldsymbol{u}),\ (\boldsymbol{v}+\boldsymbol{w}+\boldsymbol{u})]$.
(e) If $\mathbf{A}$ is symmetric and positive definite, then, so it is its inverse.
(f) Consider $\underset{m\times n}{\mathbf{A}}$ and an echelon form $\mathbf{R}$ of $\mathbf{A}$ using **row** operations, so $\mathbf{R} = \mathbf{E}\mathbf{A}$ for an invertible $\mathbf{E}$. Then, $\mathcal{C}\left(\mathbf{A}\right) = \mathcal{C}\left(\mathbf{R}\right)$.

1.9. Final June 17/18

EXERCISE 1. Consider the set $\{\boldsymbol{v}_1,\ \boldsymbol{v}_2,\ \boldsymbol{v}_3,\ \boldsymbol{v}_4\}$ of linearly independent vectors in $\mathbb{R}^4$.

(a) (0.5^{pts}) Prove that the following set $\{\boldsymbol{v}_1,\ \boldsymbol{v}_2,\ \boldsymbol{v}_3,\ (\boldsymbol{v}_1+2\boldsymbol{v}_2+\boldsymbol{v}_4)\}$ is a basis for $\mathbb{R}^4$.

(b) (0.5^{pts}) What is the dimension of the space spanned by $\{\boldsymbol{v}_1,\ (\boldsymbol{v}_1+2\boldsymbol{v}_2+\boldsymbol{v}_4)\}$. Please justify your answer.

(c) (0.5^{pts}) Prove that the set $\{(1,1,0,0),\ (0,0,0,1),\ (1,0,-1,1),\ (1,0,0,0)\}$ is a basis for $\mathbb{R}^4$ and find the third coordinate of $(1,1,1,1)$ respect to that basis.

(d) (0.5^{pts}) Find cartesian (or implicit) equations of the linear span of $\big\{(0,0,0,1),\ (1,0,-1,1)\big\}$.

(e) (0.5^{pts}) Find parametric equations of the *hiper*plane perpendicular to $(1,1,0,0)$ that goes through $(1,-1,0,1)$.

EXERCISE 2. Consider matrix $\mathbf{A}=\begin{bmatrix} a & -1 & 0\\ -1 & 3 & 1\\ 0 & 1 & 2\end{bmatrix}$

(a) (0.5^{pts}) Find an echelon form of $\mathbf{A}$ for $a=0$. Write down which elementary matrices are used in each step.

(b) (0.5^{pts}) For which values of parameter "a" matrix $\mathbf{A}$ is invertible and $\begin{pmatrix}-1\\0\\0\end{pmatrix}$ is the second column of its inverse?

(c) (0.5^{pts}) For which values of parameter "a" matrix $\mathbf{A}$ has $\lambda=3$ as an eigenvalue?

(d) (0.5^{pts}) When $a=2$ two eigenvalues are $\lambda_1=1$ and $\lambda_2=2$. Find the third eigenvalue and the corresponding eigenvectors to the third eigenvalue.

(e) (0.5^{pts}) For which values of "a" the quadratic form corresponding to $\mathbf{A}$ is positive definite?

EXERCISE 3. This question is about an m by n matrix for which

$$\mathbf{A}\boldsymbol{x}=\begin{pmatrix}1\\0\\0\end{pmatrix}\quad \text{has no solution; and}\quad \mathbf{A}\boldsymbol{x}=\begin{pmatrix}0\\1\\0\end{pmatrix}\quad \text{has exactly one solution.}$$

(a) (0.5^{pts}) Give all possible information about m and n and the rank r of $\mathbf{A}$.

(b) (0.5^{pts}) If $\mathbf{A}\boldsymbol{x}=\mathbf{0}$, which numbers could be components of $\boldsymbol{x}$.

(c) (1^{pts}) Write down an example of a matrix $\mathbf{A}$ that fits the description in this question.

(d) (0.5^{pts}) *(Not related to parts (a)–(c))* How do you know that the rank of a matrix stays the same if its first and last columns are exchanged?

MIT Course 18.06 Quiz 1, March 10, 1995

SHORT QUESTIONS SET 1. Consider the linear system $\begin{bmatrix}1 & a\\ -1 & 1\end{bmatrix}\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}b_1\\b_2\end{pmatrix}$; where a, b_1 and b_2 are parameters.

(a) Find the echelon form and discuss for which parameter values the system is solvable.

(b) For which parameter values the set of solutions is a subspace?

(c) For which values of parameter "a" the orthogonal projection of $(a,\ 1)$ onto the span of $(1,-1)$ is the zero vector?

(d) For which values of parameter "a" matrix $\mathbf{A}$ has a repeated eigenvalue? Is matrix $\mathbf{A}$ diagonalizable in this case?

SHORT QUESTIONS SET 2. *Determine whether the following statements are true or false, and explain your reasoning (to receive full credit you must explain your answers in a clear and concise way).*

(a) If the determinant of a 4×4 matrix is 4, then the rank of the matrix must be 4.

(b) If the standard vectors $\big\{\boldsymbol{e}_n,\dots,\boldsymbol{e}_n\big\}$ (the columns of the identity matrix $\mathbf{I}$) are eigenvectors of an $n\times n$ matrix, then the matrix is diagonal.

(c) If $\boldsymbol{u}\neq\boldsymbol{v}$ and both are eigenvectors of $\mathbf{A}$, then $\boldsymbol{u}$ and $\boldsymbol{v}$ are linearly independent.

(d) If $\mathbf{A}$ is an $n\times n$ matrix with fewer than n distinct eigenvalues, then $\mathbf{A}$ is not diagonalizable.

(e) If -3 is an eigenvalue of the $n\times n$ matrix $\mathbf{A}$, then there must be some vector $\boldsymbol{v}$ in $\mathbb{R}^n$ for which the equation $(\mathbf{A}+3\mathbf{I})\boldsymbol{x}=\boldsymbol{v}$ has no solution.

(f) Consider $\boldsymbol{u}=(1,0,0,0)$, $\boldsymbol{v}=(1,1,0,0)$ and consider $\mathcal{V}$, the two dimensional subspace spanned by $\{\boldsymbol{u},\boldsymbol{v}\}$, them the projection matrix of $\mathbb{R}^4$ onto $\mathcal{V}$ is $\mathbf{P}=\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix}$.

1.10. Final May 17/18

EXERCISE 1. Suppose $\mathbf{A}$ is a 2 by 2 symmetric matrix with eigenvalues 2, and 5 and corresponding eigenvectors $\boldsymbol{v}_1$, and $\boldsymbol{v}_2$.

(a) (0.5pts) Suppose $\boldsymbol{x}$ is the linear combination $\boldsymbol{x} = c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2$. Find $\mathbf{A}\boldsymbol{x}$.

(b) (0.5pts) Now take one step forward and find $\boldsymbol{x}\mathbf{A}\boldsymbol{x}$ using the symmetry of $\mathbf{A}$ (and remenber that $\boldsymbol{x} = c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2$).

(c) (0.5pts) Prove that $\boldsymbol{x}\mathbf{A}\boldsymbol{x} > 0$ for all c_1 and c_2 (except when $c_1 = c_2 = 0$).

(d) (0.5pts) Suppose those eigenvectors have length 1 (unit vectors). Show that

$$\mathbf{B} = 2 \cdot [\boldsymbol{v}_1][\boldsymbol{v}_1]^\intercal + 5 \cdot [\boldsymbol{v}_2][\boldsymbol{v}_2]^\intercal$$

has the same eigenvectors and eigenvalues as $\mathbf{A}$.

(e) (0.5pts) Is $\mathbf{B}$ necessarily the same matrix as $\mathbf{A}$ (yes or no)? Please, explain.

Based on MIT 18.06 - Quiz 3, December 5, 2005

EXERCISE 2. Consider the following vectors

$$\boldsymbol{v}_1 = \begin{pmatrix}1\\1\\0\end{pmatrix}, \qquad \boldsymbol{v}_2 = \begin{pmatrix}1\\2\\0\end{pmatrix}, \qquad \boldsymbol{v}_3 = \begin{pmatrix}0\\0\\2\end{pmatrix}.$$

We know that $\mathbf{A}\boldsymbol{v}_1 = 2\boldsymbol{v}_1$, $\mathbf{A}\boldsymbol{v}_2 = \boldsymbol{v}_2$, and $\mathbf{A}\boldsymbol{v}_3 = \boldsymbol{v}_3$, for a 3 by 3 matrix $\mathbf{A}$.

(a) (0.5pts) Solve $\mathbf{A}\boldsymbol{x} = \mathbf{0}$.

(b) (0.5pts) Find the cartesian (or implicit) equations of the eigenspace corresponding to the repeated eigenvalue.

(c) (0.5pts) Without computing $\mathbf{A}$, decide if $\mathbf{A}$ is symmetric or not. Is it diagonalizable? Justify your answer

(d) (0.5pts) Prove $B = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$ is a basis for $\mathbb{R}^3$ and find the coordinates of $\begin{pmatrix}1 & 0 & 1\end{pmatrix}$ with respect to B.

(e) (0.5pts) Find (without computing $\mathbf{A}$) the product $\mathbf{A}^3\begin{pmatrix}1\\0\\1\end{pmatrix}$.

EXERCISE 3. Consider

$$\mathbf{H} = \begin{bmatrix}1 & 1 & 0\\-1 & 1 & 0\\0 & 0 & 3\end{bmatrix}$$

(a) (0.5pts) Are the columns of $\mathbf{H}$ an orthogonal basis for $\mathbb{R}^3$? Are they an orthonormal basis for $\mathbb{R}^3$?

(b) (1pts) Find $\mathbf{H}^{-1}$ following these steps: first multiply $\mathbf{H}$ by a diagonal matrix $\mathbf{D}$ in order to get a matrix $\mathbf{Q}$ such that whose columns are an orthonormal basis of $\mathbb{R}^3$ (in other words, $\mathbf{HD} = \mathbf{Q}$ with $\mathbf{Q}$ orthogonal). Then, from the inverse of $\mathbf{HD} = \mathbf{Q}$ find an expression for $\mathbf{H}^{-1}$.

(c) (0.5pts) Find the cartesian equations for the line spanned by the first column of $\mathbf{H}$.

(d) (0.5pts) Find the projection matrix that projects $\mathbb{R}^3$ onto the subspace spanned by the first and third columns of $\mathbf{H}$.

SHORT QUESTIONS SET 1. Consider $\mathbf{A} = \begin{bmatrix}1 & 2 & 0 & 1\\0 & 0 & 1 & 1\\0 & 0 & 2 & 3\\0 & 0 & 0 & 2\end{bmatrix}$.

(a) Compute a non-null cofactor C_{ij} of matrix $\mathbf{A}$.

(b) By which elementary matrices should we multiply $\mathbf{A}$ in order to get an echelon form. Write such product of matrices in the right order.

(c) Find a basis and the dimension of the set of solutions to $\mathbf{A}\boldsymbol{x} = \mathbf{0}$.

(d) Decide if $\mathbf{A}$ is diagonalizable of not.

(e) Compute: $|\mathbf{A} - \mathbf{A}^\intercal|$.

(f) Find an orthonormal basis for the linear span of *rows* 2 and 3 of $\mathbf{A}$.

SHORT QUESTIONS SET 2. Consider $\underset{3\times3}{\mathbf{A}} = \begin{bmatrix}a_{11} & a_{12} & a_{13}\\a_{21} & a_{22} & a_{23}\\a_{31} & a_{32} & a_{33}\end{bmatrix}$ with determinant 10 and consider the matrix $\mathbf{B} = \begin{bmatrix}2a_{11} & (a_{12}+7a_{11}) & -a_{13}\\2a_{21} & (a_{22}+7a_{21}) & -a_{23}\\2a_{31} & (a_{32}+7a_{31}) & -a_{33}\end{bmatrix}$. Find:

(a) determinant of $\mathbf{B}$.
(b) determinant of $(\mathbf{A}^{-1}\mathbf{B}^{\mathsf{T}})^{-1}$.

SHORT QUESTIONS SET 3. Consider the cuadratic form $f(x, y) = ax^2 + ay^2 + 6xy$ where a is a parameter.

(a) Classify $f(x, y)$ for all values of a.
(b) Consider the matrix $\mathbf{A}$ such that $f(x, y) = f(\boldsymbol{x}) = \boldsymbol{x}\mathbf{A}\boldsymbol{x}$. Now consider the linear system $\mathbf{A}\boldsymbol{x} = \mathbf{0}$ for this matrix $\mathbf{A}$. Find the values a in order to get a system $\mathbf{A}\boldsymbol{x} = \mathbf{0}$ whose solution set consist in all points in $\mathbb{R}^2$ such that $x = y$.

1.11. Final July 16/17

EXERCISE 1. Consider a full row rank matrix $\mathbf{A}$ such that the nullspace $\mathcal{N}(\mathbf{A})$ is $\left\{ \boldsymbol{x} \in \mathbb{R}^4 : \boldsymbol{x} = a \begin{pmatrix} -1 \\ 2 \\ -3 \\ 1 \end{pmatrix}; \quad a \in \mathbb{R} \right\}$.

(a) (0.5^{pts}) What is the order of matrix $\mathbf{A}$ (Explain your answer).
(b) (1^{pts}) Give an example of such matrix $\mathbf{A}$.
(c) (0.5^{pts}) For which right hand side vectors $\boldsymbol{b}$ the system $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$ is not solvable?
(d) (0.5^{pts}) Rows of $\mathbf{A}$ belong to $\mathbb{R}^n$ (that is $\mathcal{C}(\mathbf{A}^\intercal) \subset \mathbb{R}^n$). Are there vectors in $\mathbb{R}^n$ that do not belong to $\mathcal{C}(\mathbf{A}^\intercal)$? If the answer is "yes", provide an example.

Ejercicio propuesto por Haydee Lugo

EXERCISE 2. Consider the set $B = \{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3, \boldsymbol{u}_4\}$, where

$$\boldsymbol{u}_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \ \boldsymbol{u}_2 = \begin{pmatrix} 1 \\ -2 \\ 0 \\ 0 \end{pmatrix}, \ \boldsymbol{u}_3 = \begin{pmatrix} 0 \\ 0 \\ -2 \\ 4 \end{pmatrix} \quad \text{and} \quad \boldsymbol{u}_4 = \begin{pmatrix} 0 \\ 0 \\ -2 \\ -3 \end{pmatrix},$$

and also consider $\underset{4\times 4}{\mathbf{A}}$ such that

$$\mathbf{A}\boldsymbol{u}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{A}\boldsymbol{u}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{A}\boldsymbol{u}_3 = \begin{pmatrix} 0 \\ 0 \\ 2 \\ 0 \end{pmatrix} \quad \text{and} \quad \mathbf{A}\boldsymbol{u}_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ -1 \end{pmatrix}.$$

(a) (0.5^{pts}) Prove B is a basis for $\mathbb{R}^4$.
(b) (1^{pts}) Consider $\mathbf{B} = \begin{bmatrix} \boldsymbol{u}_1, & \boldsymbol{u}_2, & \boldsymbol{u}_3, & \boldsymbol{u}_4 \end{bmatrix}$, a matrix whose columns are the vectors in B. Solve by gaussian elimination $\mathbf{B}\boldsymbol{x} = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}$. Write the coordinates of $\begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}$ with respect to basis B.
(c) (0.5^{pts}) Check that $\mathbf{AB}$ is a diagonal matrix. Use this result in order to prove $\mathbf{A}$ is invertible and find $\mathbf{A}^{-1}$.
(d) (0.5^{pts}) Consider the following quadratic form $f(\boldsymbol{x}) = \boldsymbol{x}\mathbf{A}\boldsymbol{x}$. Compute $f(\boldsymbol{u}_1)$ and $f(\boldsymbol{u}_4)$. Can you classify $f(\boldsymbol{x})$ using this information? (Explain your answer)

EXERCISE 3. This question is about the matrix $\mathbf{A} = \mathbf{B} + b\mathbf{I}$ where $\mathbf{B}$ is the all-ones matrix:

$$\mathbf{A} = \mathbf{B} + b \cdot \mathbf{I} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} + b \cdot \mathbf{I} = \begin{bmatrix} 1+b & 1 & 1 & 1 \\ 1 & 1+b & 1 & 1 \\ 1 & 1 & 1+b & 1 \\ 1 & 1 & 1 & 1+b \end{bmatrix}.$$

(a) (0.5^{pts}) Check that $\boldsymbol{v} = \begin{pmatrix} 1 & 1 & 1 & 1 \end{pmatrix}$ is an eigenvector for $\mathbf{B}$. Find the eigenvalues of $\mathbf{B}$ (remember that $\mathbf{B}$ is the matrix full of ones).
(b) (0.5^{pts}) What are the eigenvalues of $\mathbf{A}$?
Hint: what is $(\mathbf{B} + b\mathbf{I})\boldsymbol{x}$ when $\boldsymbol{x}$ and λ are an eigenvector and its correspondant eigenvalue of $\mathbf{B}$?
(c) (0.5^{pts}) If $b = 2$, what is the determinant of $\mathbf{A}$?
(d) (0.5^{pts}) Suppose you know that $\boldsymbol{x}\mathbf{A}\boldsymbol{x} > 0$ for every nonzero vector $\boldsymbol{x}$ (same matrix $\mathbf{A}$.) What are the possible values of b?
(e) (0.5^{pts}) When $b = 1$ the inverse matrix of $\mathbf{A}$ has the form $\mathbf{A}^{-1} = \mathbf{I} + c\mathbf{B}$. Figure out $\mathbf{B}^2$ and then choose the number c so that $\mathbf{AA}^{-1} = \mathbf{I}$.

Basado en MIT 18.06 - Final Exam, December 19, 2005

SHORT QUESTIONS SET 1. *Parts a) and b) of this set are true/false type questions. Parts c) and d) ask you to prove something.*

Determine whether the following statements are true or false, and explain your reasoning (to receive full credit you must explain your answers in a clear and concise way).

Let the set $B = \{\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}\}$ be a basis for the subspace $\mathcal{S}$ in $\mathbb{R}^4$.

(a) Vector $\boldsymbol{u}-\boldsymbol{v}$ has coordinates $(1,-1)$ with respecto to basis B.
(b) Vector $\boldsymbol{u}+3\boldsymbol{v}$ belongs to $\mathcal{S}$.

For parts c) and d) consider that B is also orthonormal. *Then, if* $\mathbf{X}=\begin{bmatrix}\boldsymbol{u}, & \boldsymbol{v}, & \boldsymbol{w}\end{bmatrix}$ *is the matrix whose columns are the vectors in B, the matrix $\mathbf{P}=\mathbf{X}\mathbf{X}^{\intercal}$ projects $\mathbb{R}^4$ onto S.*

(c) **Prove** matrix $\mathbf{P}$ is symmetric and idempotent.
(d) Using $\mathbf{P}$ is symmetric and idempotent, prove that $\mathbf{P}\boldsymbol{y}$ is orthogonal to $(\boldsymbol{y}-\mathbf{P}\boldsymbol{y})$ for all $\boldsymbol{y}\in\mathbb{R}^4$.

Short questions set 2. Consider

$$\mathbf{A}=\begin{bmatrix}1 & 0 & 1 & 0\\ -1 & 1 & 0 & 0\\ 0 & 0 & 1 & 1\\ 0 & a & 0 & -1\end{bmatrix}$$

Find all values for a such that:
(a) There exists $\mathbf{A}^{-1}$
(b) Determinant of $\left[(\mathbf{A}^{\intercal}\mathbf{A})^{-1}\mathbf{A}^{\intercal}\right]$ equals $\frac{1}{4}$.
(c) Columns of $\mathbf{A}$ span a three dimensional space.

Short questions set 3. Given the matrix $\mathbf{A}=\begin{bmatrix}0 & 3 & 0\\ a & 0 & b\\ 0 & 4 & 0\end{bmatrix}$.

(a) Find a and b such that there are matrices $\mathbf{P}$ (invertible) and $\mathbf{D}$ (diagonal and real) and $\mathbf{AP}=\mathbf{PD}$.
(b) Consider $a=3$ and $b=4$. Classify the quadratic form $\boldsymbol{x}\mathbf{A}\boldsymbol{x}$.
(c) Consider the case in which orthogonal eigenvectors can be found for $\mathbf{A}$ and find a vector $\boldsymbol{v}$ with length 1 such that $\mathbf{A}\boldsymbol{v}$ has length 5 and it is pointing to the same place as $\boldsymbol{v}$ (the same direction of $\boldsymbol{v}$).

1.12. Final May 16/17

Exercise 1. Consider the following vectors in $\mathbb{R}^4$: $\boldsymbol{v}_1 = \begin{pmatrix}1\\1\\0\\a\end{pmatrix}$, $\boldsymbol{v}_2 = \begin{pmatrix}1\\0\\1\\0\end{pmatrix}$, $\boldsymbol{v}_3 = \begin{pmatrix}0\\0\\1\\1\end{pmatrix}$, $\boldsymbol{v}_4 = \begin{pmatrix}0\\1\\0\\b\end{pmatrix}$,

where a and b are parameters.

(a) (0.5^{pts}) Find the values a and b such that those vectors form a basis for $\mathbb{R}^4$.

(b) (1^{pts}) Consider the matrix $\mathbf{A} = \begin{bmatrix}\boldsymbol{v}_1, & \boldsymbol{v}_2, & \boldsymbol{v}_3, & \boldsymbol{v}_4\end{bmatrix}$, whose columns are the given vectors. Find $\mathbf{A}^{-1}$ when $a = b$ or explain why it is impossible.

For the next two questions consider $a = 0$ and $b = 1$, and also $\mathcal{S} = \mathcal{L}(\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3, \boldsymbol{v}_4)$, i.e., the spam of the four vectors.

(c) (0.5^{pts}) Find the dimension and a basis for $\mathcal{S}$.

(d) (0.5^{pts}) Compute the coordinates of $\boldsymbol{b} = \begin{pmatrix}1 & \frac{1}{2} & \frac{1}{2} & 0\end{pmatrix}$ with respect to $\boldsymbol{v}_1$, $\boldsymbol{v}_2$ and $\boldsymbol{v}_3$.

Exercise 2. This 4 by 4 matrix H is a Hadamard matrix:

$$\mathbf{H} = \begin{bmatrix}1 & 1 & 1 & 1\\ 1 & -1 & 1 & -1\\ 1 & 1 & -1 & -1\\ 1 & -1 & -1 & 1\end{bmatrix} \qquad \text{Key Properties: } \begin{cases}\mathbf{H}^{\mathsf{T}} = \mathbf{H}\\ \mathbf{H}^2 = 4\mathbf{I}\end{cases}$$

Hint. *Don't work too much!*
For a) and b) it is better to use the fact that $\mathbf{H}^2 = \mathbf{H}\mathbf{H} = 4\mathbf{I}$.
To solve part c) note that $\mathbf{H}^{\mathsf{T}} = \mathbf{H}$.

(a) (0.5^{pts}) Figure out the eigenvalues of $\mathbf{H}$. Explain your reasoning.

(b) (1^{pts}) Figure out $\mathbf{H}^{-1}$ and the determinant of $\mathbf{H}$. Explain your reasoning.

(c) (1^{pts}) This matrix $\mathbf{S}$ contains three eigenvectors of $\mathbf{H}$. Find a 4th eigenvector $\boldsymbol{v}_4$ and explain your reasoning:

$$\mathbf{S} = \begin{bmatrix}1 & 1 & 0\\ 1 & 0 & -1\\ 1 & 0 & 1\\ -1 & 1 & 0\end{bmatrix}.$$

MIT 18.06 - Quiz 3, December 5, 2005

Exercise 3. The set of solutions to $\underset{3\times 3}{\mathbf{A}}\,\boldsymbol{x} = \begin{pmatrix}2\\4\\6\end{pmatrix}$ is

$$S = \left\{ \boldsymbol{x} \,\middle|\, \boldsymbol{x} = \begin{pmatrix}1\\2\\3\end{pmatrix} + \alpha\begin{pmatrix}0\\1\\1\end{pmatrix} + \beta\begin{pmatrix}1\\1\\0\end{pmatrix}; \quad \forall \alpha, \beta \in \mathbb{R} \right\}$$

(**Note:** *Don't work too much! To answer the following questions you don't need to known* $\mathbf{A}$.)

(a) (0.5^{pts}) Prove the following statements:

1. vector $\boldsymbol{y} = \begin{pmatrix}1 & 2 & 3\end{pmatrix}$ is a solution to the linear system
2. the set $\mathcal{N} = \left\{ \boldsymbol{z} \,\middle|\, \boldsymbol{z} = \alpha\begin{pmatrix}0\\1\\1\end{pmatrix} + \beta\begin{pmatrix}1\\1\\0\end{pmatrix}; \quad \forall \alpha, \beta \in \mathbb{R} \right\}$ is the solution to the system $\mathbf{A}\boldsymbol{x} = \mathbf{0}$.

(b) (0.5^{pts}) Find the cartesian equations for the following subspace $\left\{\boldsymbol{x} \in \mathbb{R}^3 | \mathbf{A}\boldsymbol{x} = \mathbf{0}\right\}$.

(c) (0.5^{pts}) Choose a basis for $\mathcal{N} = \left\{\boldsymbol{x} \in \mathbb{R}^3 | \mathbf{A}\boldsymbol{x} = \mathbf{0}\right\}$ and find the projection matrix $\mathbf{P}$ that projects $\mathbb{R}^3$ onto $\mathcal{N}$.

(d) (1^{pts}) Is $\boldsymbol{d} = \begin{pmatrix}3 & 2 & 4\end{pmatrix}$ in $\mathcal{N} = \left\{\boldsymbol{x} \in \mathbb{R}^3 | \mathbf{A}\boldsymbol{x} = \mathbf{0}\right\}$? If it belongs to $\mathcal{N}$ compute its coordinates with respect to the basis used in part c); otherwise find the closest vector in $\mathcal{N}$.

SHORT QUESTIONS SET 1. *Determine whether the following statements are true or false, and explain your reasoning (to receive full credit you must explain your answers in a clear and concise way).*

(a) If $\underset{n\times n}{\mathbf{Q}}$ is a symmetric matrix whose columns are orthonormal eigenvectors of $\underset{n\times n}{\mathbf{A}}$, then $\mathbf{A}$ is symmetric.

(b) If $\mathbf{A}$ is a symmetric matrix with $a_{11} > 0$ and $\det \mathbf{A} < 0$, then the associted quadratic form is not positive definte nor negative definite.

(c) Consider a matrix of order 3 $\mathbf{A}$ and $\mathcal{S}_{\lambda=1} = \mathcal{L}\left\{\begin{pmatrix}2\\1\\0\end{pmatrix}, \begin{pmatrix}0\\1\\1\end{pmatrix}\right\}$ and $\mathcal{S}_{\beta=2} = \mathcal{L}\left\{\begin{pmatrix}1\\-2\\2\end{pmatrix}\right\}$ the eigenspaces associated to its eigenvalues $\lambda = 1$ and $\beta = 2$. Then $\mathbf{A}$ is symetric.

(d) If $\mathbf{A}$ is symmetric and invertible, then the associated quadratic form $\boldsymbol{x}\mathbf{A}^2\boldsymbol{x}$ is positive definite.

SHORT QUESTIONS SET 2. Consider $\mathbf{A} = \begin{bmatrix}0&0&1&0\\1&1&0&0\\1&2&1&3\\1&3&1&9\end{bmatrix}$.

(a) Find the determinants of $\mathbf{A}$ and $\mathbf{A}^{-1}$.

(b) Find the $(1, 2)$ entry of $\mathbf{A}^{-1}$.

Basado en MIT 18.06 - Quiz 2, April 1, 2005

SHORT QUESTIONS SET 3.

(a) Prove that if $\mathbf{M}$ is idempotent and invertible, then $\mathbf{M}$ is the identity matrix.

(b) The characteristic polynomial of $\mathbf{N}$ is $P(\lambda) = \lambda^4 - 3\lambda^3 + 2\lambda^2$. Find the eigenvalues. Are we sure $\mathbf{N}$ is diagonalizable?

(c) Consider matrix $\mathbf{B} = \mathbf{A}\mathbf{A}^{\intercal}$. Prove that $\mathbf{B}$ is symmetric.

(d) Suppose the characteristic polynomial of the above matrix $\mathbf{B}$ is $P(\lambda) = \lambda(\lambda - 2)^2(\lambda - 4)$. Prove that rank of $(\mathbf{B} - 2\mathbf{I})$ is two.

1.13. Final June 15/16

Exercise 1. Consider the following system of equations, $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$:

$$\begin{cases} 2x_1 + x_2 + x_3 = 2\alpha \\ 4x_1 + 2x_2 + 2x_3 = 4\alpha \\ 6x_1 + 2x_2 + 3x_3 = 0 \end{cases}$$

(a) (1^{pts}) What conditions on α make the system solvable?
(b) (1^{pts}) Solve the system in that case.
(c) (0.5^{pts}) How do you known, without computing the determinant, that $\det \mathbf{A}$ (the determinant of the coeficient matrix of the system) is zero? (You don't have to compute the determinant, just only answer why it is zero, using the former results).

Exercise 2. Let

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 0 \\ 2 & 1 & 2 \\ 2 & 0 & 3 \end{bmatrix}.$$

(a) (0.5^{pts}) Compute $\det(\mathbf{A})$.
(b) (1^{pts}) Find $\mathbf{A}^{-1}$.
(c) (0.5^{pts}) For the same matrix $\mathbf{A}$:

- Is the system $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$ solvable for any $\boldsymbol{b} \in \mathbb{R}^3$?
- Could $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$ have no solutions for some $\boldsymbol{b} \in \mathbb{R}^3$, but infinite solutions for a different $\boldsymbol{b} \in \mathbb{R}^3$?

(d) (0.5^{pts}) For the same matrix $\mathbf{A}$, find a vector $\boldsymbol{b}$ such that the solution to $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$ will be $x_1 = 1$, $x_2 = 2$, $x_3 = -1$.

Exercise 3.

(a) (0.5^{pts}) Let $\mathbf{A}$ be the 5 by 5 matrix $\mathbf{A} = \begin{bmatrix} 2 & 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 & 1 \\ 1 & 1 & 2 & 1 & 1 \\ 1 & 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 1 & 2 \end{bmatrix}$. Is $\mathbf{A}$ diagonalizable? Explain your answer.
(b) (1^{pts}) Find all five eigenvalues of $\mathbf{A}$ by noticing that $\mathbf{A}-\mathbf{I}$ has rank 1 and the trace of $\mathbf{A}$ is ____________.
(c) (1^{pts}) Find five linear independent eigenvectors of $\mathbf{A}$.

Basado en MIT Course 18.06. Final Exam. Professor Strang. May 16, 2005

Short questions set 1.
(a) Find a system fo parametric representation for the line passing through the point $\boldsymbol{p} = (1, -3, 1)$ and it is perpendicular to the plane spaned by $\boldsymbol{u} = (7, 3, 0)$ and $\boldsymbol{v} = (4, 0, 3)$.
(b) Find a parametric representation for the same line.

Short questions set 2. Prove the following statement:

If $\{\boldsymbol{u}_1,\ \boldsymbol{u}_2,\ \boldsymbol{u}_3\}$ is an orthogonal set, then these three vectors are linearly independent.

Short questions set 3. *Determine whether the following statements are true or false, and explain your reasoning (to receive full credit you must explain your answers in a clear and concise way).*
(a) Any three vectors in $\mathbb{R}^3$ span $\mathbb{R}^3$.
(b) The columns of a matrix are linearly independent if and only if its rank equals the number of columns.
(c) If the determinant of a squared matrix $\mathbf{A}$ is 1 or -1, then $\mathbf{A}$ must be an orthogonal (orthonormal) matrix.
(d) If $\mathbf{A}$ is an orthogonal (orthonormal) matrix, its determinant must be 1 or -1.
(e) If a 10×10 matrix $\mathbf{A}$ has 6 distinct eigenvalues, then, the rank must be at least 5.

Short questions set 4. Consider the following linear system $\mathbf{A}\boldsymbol{x} = \mathbf{0}$ where

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 1 & 2 & 3 \\ a & 1 & 1 & 2 \end{bmatrix}.$$

(a) Find the values of a such as the set of solutions of the linear system is a line.
(b) Find the values of a such as the set of solutions of the linear system is a plane?

1.14. Final May 15/16

EXERCISE 1. Consider the following points in $\mathbb{R}^3$, $\boldsymbol{a} = (1, 0, 3)$ and $\boldsymbol{b} = (-\frac{1}{3}, 0, -1)$.

(a) (0.5^{pts}) Find a parametric equation of the line that goes through $\boldsymbol{a}$ and $\boldsymbol{b}$.
(b) (0.5^{pts}) Find a cartesian (or implicit) system of equations of the line that goes through $\boldsymbol{a}$ and $\boldsymbol{b}$.
(c) (0.5^{pts}) Is that line a subspace in $\mathbb{R}^3$? Explain your answer.
(d) (0.5^{pts}) Write the proyection matrix that projects any point in $\mathbb{R}^3$ on the line that goes through $\boldsymbol{a}$ and $\boldsymbol{b}$.
(e) (0.5^{pts}) On that line, find the closest point to $\boldsymbol{z} = (2, 2, 2)$.

Ejercicio propuesto por Rafael Lopez.

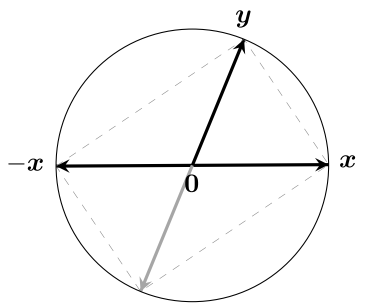
EXERCISE 2. Consider matrix $\mathbf{A} = \begin{bmatrix} 2 & 1-m & 0 \\ 1-m & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix}$.

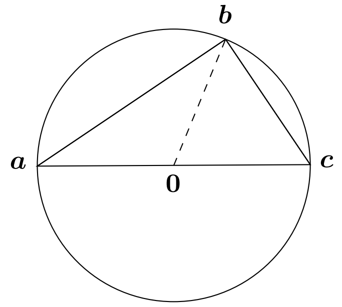
(a) (0.5^{pts}) Find $\det(\mathbf{A})$ as a function of m. For which m is $\mathbf{A}$ a singular matrix?
(b) (1^{pts}) By tranformations of $\mathbf{A}$, find two 3 by 3 matrices $\mathbf{B}$ and $\mathbf{C}$, such that $|\mathbf{B}| = -|\mathbf{A}|$ and $|\mathbf{C}| = \frac{1}{2}|\mathbf{A}|$.
(c) (1^{pts}) For $m = 1$, **and using the Cramer rule**, find the solution to $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$, where $\boldsymbol{b} = \begin{bmatrix} 3 & 4 & 2 \end{bmatrix}^{\mathsf{T}}$.

Ejercicio propuesto por Haydee Lugo

EXERCISE 3.

(a) (1^{pts}) Consider two vectors $\boldsymbol{x}$ and $\boldsymbol{y}$ in $\mathbb{R}^2$ with the same length, ie., $\|\boldsymbol{x}\| = \|\boldsymbol{y}\|$. Prove that $\boldsymbol{y} + \boldsymbol{x}$ and $\boldsymbol{y} - \boldsymbol{x}$ are orthogonal vectors.
(b) (0.5^{pts}) Draw vectors $\boldsymbol{y} + \boldsymbol{x}$ and $\boldsymbol{y} - \boldsymbol{x}$ in the figure on the left.





(c) (1^{pts}) Prove that segments $[\boldsymbol{a} \leftrightarrow \boldsymbol{b}]$ and $[\boldsymbol{b} \leftrightarrow \boldsymbol{c}]$, of the triangle in the figure on the right, are perpendicular.

SHORT QUESTIONS SET 1. Consider $\mathbf{C}$, a symmetric and positive defined $n \times n$ matrix. If $\mathbf{M} = \mathbf{A}^{\mathsf{T}}\mathbf{C}\mathbf{A}$, where $\mathbf{A}$ is $n \times m$:

(a) Prove that $\mathbf{M}$ is symmetric.
(b) Prove that $\boldsymbol{x}\mathbf{M}\boldsymbol{x} \geq 0$ for all $\boldsymbol{x} \in \mathbb{R}^m$.
(c) If $\boldsymbol{x}\mathbf{M}\boldsymbol{x} = 0$ for some $\boldsymbol{x} \neq \mathbf{0}$ ¿what is the smallest eigenvalue? Explain your answer

Pregunta propuesta por Manuel Morán

SHORT QUESTIONS SET 2. Consider $\mathbf{A} = \begin{bmatrix} 2 & 4 & 2 \\ 0 & 4 & 2 \\ 0 & 0 & 2 \end{bmatrix}$

(a) Is $\mathbf{A}$ a diagonalizable matrix?
(b) Is $\mathbf{A}$ an invertible matrix? If yes, find $\mathbf{A}^{-1}$.

Pregunta propuesta por Haydee Lugo

SHORT QUESTIONS SET 3. *Determine whether the following statements are true or false, and explain your reasoning (to receive full credit you must explain your answers in a clear and concise way).*

(a) If $\mathcal{V} = span(\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_k)$, then $\dim(\mathcal{V}) \leq k$.
(b) If $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_k$ are linearly independent vectors in $\mathcal{V}$, then $\dim(\mathcal{V}) \geq k$.
(c) A system of three equations in four unknowns cannot have a unique solution.
(d) A system of four equations in three unknowns cannot have more than one solution.
(e) The scalar product (dot product) of two vectors in $\mathbb{R}^3$ is a vector in $\mathbb{R}^3$.

1.15. Final June 14/15

EXERCISE 1. Consider a linear system of algebraic equation $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$. Here the matrix $\mathbf{A}$ has three rows and four columns.

(a) (0.5^{pts}) Does such a linear system always have at least one solution? If not provide an example for which no solution exists.
(b) (0.5^{pts}) Can such a linear system have a unique solution? If so, provide and example of a problem with this property.
(c) (0.5^{pts}) Formulate, if possible, necessary and sufficient conditions on $\mathbf{A}$ and $\boldsymbol{b}$ which guarantee that at least one solution exists.
(d) (0.5^{pts}) Formulate, if possible, necessary and sufficient conditions on $\mathbf{A}$ which guarantee that at least one solution exists for any choice of $\boldsymbol{b}$.
(e) (0.5^{pts}) Now, consider the system $\mathbf{A}^{\mathsf{T}}\boldsymbol{y} = \boldsymbol{c}$. Can such a linear system have an infinite number of solutions? If so, provide and example of a problem with this property.

EXERCISE 2. Suppose the 3 by 3 matrix $\mathbf{A}$ has the following property $\mathcal{Z}$: *Along each of its rows, the entries add up to zero.*

(a) (0.5^{pts}) Find a nonzero solution to $\mathbf{A}\boldsymbol{x} = \mathbf{0}$.
(b) (1^{pts}) Prove that $\mathbf{A}^2$ also has property $\mathcal{Z}$.
(c) (0.5^{pts}) What can you say about the dimension of the set of solutions to $\mathbf{A}^{\mathsf{T}}\boldsymbol{x} = \mathbf{0}$ and why?
(d) (0.5^{pts}) Find an eigenvalue of the matrix $\mathbf{A}^3$.

Basado en MIT Course 18.06 Ejercicio 9 Final Spring 1999

EXERCISE 3. Let

$$\mathbf{A} = \begin{bmatrix} 3 & 4 & 6 \\ 0 & 1 & 0 \\ -1 & -2 & -2 \end{bmatrix}.$$

(a) (0.5^{pts}) Find the eigenvalues of the singular matrix $\mathbf{A}$.
(b) (1^{pts}) Find a basis of $\mathbb{R}^3$ consisting of eigenvectors of $\mathbf{A}$.
(c) (1^{pts}) Compute

$$\mathbf{A}^{99}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

Compute it by expressing $(1;1;1)$ as a combination of eigenvectors or by diagonalizing $\mathbf{A} = \mathbf{S}\mathbf{D}\mathbf{S}^{-1}$.

MIT Course 18.06 Quiz 2, Ejercicio 2 Final Fall 1999

SHORT QUESTIONS SET 1. This problem is about the matrices

$$\mathbf{A} = \begin{bmatrix} \sqrt{2} & 1 \\ 0 & \sqrt{2} \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} 2 & 1 & b \\ 1 & 2 & 1 \\ b & 1 & 2 \end{bmatrix}.$$

(a) (0.5^{pts}) Exactly why is it imposible to diagonalize $\mathbf{A}$ in the form $\mathbf{A} = \mathbf{S}\mathbf{D}\mathbf{S}^{-1}$?
(b) (0.5^{pts}) Find *all* eigenvectors of $\mathbf{A}$.
(c) (0.5^{pts}) Discuss whether the matrix $\mathbf{B}$ is definite, semidefinite or not definite depending on the values of b.

Basado en MIT Course 18.06 Quiz 3. May 6, 2011

SHORT QUESTIONS SET 2. Consider the matrix:

$$\mathbf{A} = \begin{bmatrix} 1 & c & 2 \\ c & 2 & c \\ 1 & c & 1 \end{bmatrix}.$$

(a) (0.5^{pts}) For which values of c the determinant of $\mathbf{A}$ is 0.
(b) (0.5^{pts}) Find the inverse of $\mathbf{A}$ when $c = 0$.
(c) (0.5^{pts}) For $c = 1$, solve the linear system $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$, where $\boldsymbol{b} = \begin{pmatrix} 4 \\ 1 \\ 2 \end{pmatrix}$.

Short questions set 3. *Determine whether the following statements are true or false, and explain your reasoning (to receive full credit you must explain your answers in a clear and concise way).*

(a) (0.5^{pts}) If $\mathbf{A}$ is symetric, then so it is $\mathbf{A}^2$.
(b) (0.5^{pts}) If $\mathbf{A}^2$ is symetric, then so it is $\mathbf{A}$.
(c) (0.5^{pts}) If $\lambda = 0$ is eigenvector of $\mathbf{A}$ then the system $\mathbf{A}\boldsymbol{x} = \mathbf{0}$ has a sole solution.
(d) (0.5^{pts}) If matrix $\mathbf{A}$ is invertible, then $\mathbf{A}$ is diagonalizable.

1.16. Final May 14/15

EXERCISE 1.

Suppose $\mathbf{A}$ is a 5 by 7 matrix, and $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$ has a solution for every right side $\boldsymbol{b}$. Then,

(Your answers could refer to dimension/basis/linear independence/spanning a space/$\mathbb{R}^n$ ...; and you must justify your answer.)

(a) (0.5^{pts}) What do we know about the column space of $\mathbf{A}$?
(b) (0.5^{pts}) What do we know about the rows of $\mathbf{A}$? (are they dependent or independent?)
(c) (0.5^{pts}) What do we know about the nullspace of $\mathbf{A}$?
(d) (0.5^{pts}) What do we know about the left nullspace of $\mathbf{A}$?
(e) (0.5^{pts}) True or false (with reason):

The columns of $\mathbf{A}$ are a basis for the column space of $\mathbf{A}$.

Basado en Ejercicio 2 Final Spring 1999

EXERCISE 2.

Suppose $\mathbf{A}$ has eigenvalues $\lambda_1 = 3$, $\lambda_2 = 1$, and $\lambda_3 = 0$ with corresponding eigenvectors

$$\boldsymbol{x}_1 = \begin{pmatrix}1\\1\\1\end{pmatrix}; \quad \boldsymbol{x}_2 = \begin{pmatrix}0\\1\\1\end{pmatrix}; \quad \boldsymbol{x}_3 = \begin{pmatrix}0\\0\\1\end{pmatrix}.$$

(a) (0.5^{pts}) Before computing $\mathbf{A}$, how do you know that the third column of $\mathbf{A}$ contains all zeros?
(b) (1^{pts}) Find the matrix $\mathbf{A}$.
(c) (1^{pts}) By transposing $\mathbf{S}^{-1}\mathbf{A}\mathbf{S} = \mathbf{D}$, find the eigenvectors $\boldsymbol{y}_1$; $\boldsymbol{y}_2$ and $\boldsymbol{y}_3$ of $\mathbf{A}^{\intercal}$.

Ejercicio 4 Final Spring 1999

EXERCISE 3. Consider the linear system $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$, where $\mathbf{A} = \begin{bmatrix}1&2&1\\2&1&2\\1&a&c\end{bmatrix}$ and $\boldsymbol{b} = \begin{pmatrix}0\\a\\0\end{pmatrix}$. Find (if it is possible) values for a and c such that:

(a) (0.5^{pts}) There is more than one linear combination of columns of $\mathbf{A}$ that is equal to $\boldsymbol{b}$.
(b) (0.5^{pts}) The system has no solution.
(c) (0.5^{pts}) There are only two free variables.
(d) (0.5^{pts}) All columns are pivot columns.
(e) (0.5^{pts}) The second variable x_2 is free (i.e., the second column has no pivot).

Ejercicio propuesto por Rafael Lopez

SHORT QUESTIONS SET 1. Consider the matrices

$$\mathbf{A} = \begin{bmatrix}x&2&3\\-x&x&0\\3&2&5\end{bmatrix}; \qquad \mathbf{B} = \begin{bmatrix}x&1&1\\1&x&1\\1&1&1\end{bmatrix};$$

(a) (0.5^{pts}) What value(s) of x give $\det \mathbf{A} = 0$
(b) (0.5^{pts}) What value(s) of x give positive definite matrix $\mathbf{B}$.
(c) (0.5^{pts}) What value(s) of x give negative definite matrix $\mathbf{B}$.

basado en MIT Course 18.06 Quiz 2, April 10, 1996

SHORT QUESTIONS SET 2. *Determine whether the following statements are true or false, and explain your reasoning (to receive full credit you must explain your answers in a clear and concise way).*

(a) (0.5^{pts}) If the characteristic polinomial of a 7×7 matrix $\mathbf{A}$ is $f_{\mathbf{A}}(\lambda) = \lambda(\lambda^2 - 1)(\lambda^2 - 2)(\lambda^2 - 3)$, then $\mathbf{A}$ must be diagonalizable.
(b) (0.5^{pts}) Suppose you know that -3, 2, 7 are eigenvalues of a 5×5 matrix $\mathbf{A}$. If any of these eigenvalues has a 3-dimensional eigenspace, then $\mathbf{A}$ must be invertible.
(c) (0.5^{pts}) If the product of two squared matrices $\mathbf{A}$ and $\mathbf{B}$ is invertible, then $\mathbf{A}$ must be invertible as well.

SHORT QUESTIONS SET 3. Consider the line passing through the points $(2, 4, 1)$ and $(1, 3, 1)$.

(a) (0.5^{pts}) Find a parametric representation for that line.
(b) (0.5^{pts}) Find a implicit representation for the same line.

SHORT QUESTIONS SET 4. Consider the set $\mathcal{W} = \left\{ (\boldsymbol{x}_4, \ldots, \boldsymbol{x}_n) \in \mathbb{R}^4 \text{ such that } x_4 = bx_1 \right\}$

(a) (0.5^{pts}) For which values b the set $\mathcal{W}$ is a subspace of $\mathbb{R}^4$.

(b) (0.5^{pts}) For $b = 1$, find the dimensión and write a basis of $\mathcal{W}$.

Ejercicio propuesto por Rafael Lopez

1.17. Final July 13/14

Exercise 1. Consider the quadratic form $q(x,y,z) = ax^2 + 4y^2 - 2z^2 + 8yz$:

(a) (0.5^{pts}) Classify the quadratic form in terms of the parameter a.

For questions (b), (c) and (d) consider $a = 0$

(b) (0.5^{pts}) Find the eigenvalues of the matrix $\mathbf{A}$ corresponding to the quadratic form $q(x,y,z)$.
(c) (0.5^{pts}) Find three linearly independent eigenvectors for $\mathbf{A}$.
(d) (0.5^{pts}) Find a diagonal matrix $\mathbf{D}$ and an orthonormal matrix $\mathbf{Q}$ such that $\mathbf{A} = \mathbf{QDQ}^{\intercal}$
(e) (0.5^{pts}) Has the quadratic form $q(x,y,z)$ a minimun at $(x,y,z) = (0,0,0)$? Explain your answer.

Variación de un ejercicio propuesto por Maria Jesus Moreta

Exercise 2. Suppose $\mathbf{A}$ is a real m by n matrix.

(a) (1^{pts}) Prove that the simetric matrix $\mathbf{A}^{\intercal}\mathbf{A}$ has the property $\boldsymbol{x}\mathbf{A}^{\intercal}\mathbf{A}\boldsymbol{x} \geq 0$, for every vector $\boldsymbol{x}$ in $\mathbb{R}^n$. Explain each step in your reasoning.
(b) (1^{pts}) According to part (a), the matrix $\mathbf{A}^{\intercal}\mathbf{A}$ is positive *semi*definite at least — and possibly positive definite. Under what condition on $\mathbf{A}$ is $\mathbf{A}^{\intercal}\mathbf{A}$ positive definite?
(c) (0.5^{pts}) If $m < n$ prove that $\mathbf{A}^{\intercal}\mathbf{A}$ is not positive definite.

MIT Course 18.06 Quiz 3. May 6, 2011

Exercise 3.

Consider $\mathcal{S} = \mathrm{L}(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w})$, the subspace spaned by $\boldsymbol{u} = \begin{pmatrix}2\\0\\1\\2\end{pmatrix}$, $\boldsymbol{v} = \begin{pmatrix}1\\0\\0\\1\end{pmatrix}$ and $\boldsymbol{w} = \begin{pmatrix}3\\0\\1\\3\end{pmatrix}$.

(a) (0.5^{pts}) Find a basis for $\mathcal{S}$
(b) (0.5^{pts}) Is vector $(1,0,-1,1)$ in $\mathcal{S}$? If the answer is yes, find its coordinates with respect to the basis from part (a); in other words, write $(1,0,-1,1)$ as a linear combination of vectors of the basis from part (a).
(c) (0.5^{pts}) Find implicit (or cartesian) equations of $\mathcal{S}$.
(d) (0.5^{pts}) Find a basis for the orthogonal complement $\mathcal{S}^{\perp}$ of the subspace of $\mathcal{S}$.
(e) (0.5^{pts}) Find a vector $\boldsymbol{z}$ such that the matrix with columns $[\boldsymbol{u},\boldsymbol{v},\boldsymbol{w},\boldsymbol{z}]$ has rank 3.

Short questions set 1. True or false (to receive full credit you must explain your answers in a clear and concise way)

Consider two n by n matrices $\mathbf{A}$ and $\mathbf{B}$ such that $\mathbf{A}^2 = \mathbf{I}$ and $\mathbf{B}^2 = \mathbf{B}$, then:

(a) (0.5^{pts}) $\mathbf{A} = \mathbf{A}^{-1}$.
(b) (0.5^{pts}) If $\mathbf{A}$ is symmetric, then $\mathbf{A}$ is orthonormal.
(c) (0.5^{pts}) Rank of $\mathbf{B}$ is n.
(d) (0.5^{pts}) $\mathbf{B}^2 = \mathbf{B} \implies \mathbf{B}\cdot\mathbf{B} = \mathbf{B} \implies \mathbf{B}\cdot\mathbf{B}\cdot\mathbf{B}^{-1} = \mathbf{B}\cdot\mathbf{B}^{-1} \implies \mathbf{B} = \mathbf{I}$.

Short questions set 2. True or false (to receive full credit you must explain your answers in a clear and concise way)

(a) (0.5^{pts}) If 0 is an eigenvalue of an $n \times n$ matrix $\mathbf{A}$, then $\operatorname{rg}\left(\mathbf{A}\right) < n$.
(b) (0.5^{pts}) If -3 is an eigenvalue of the $n \times n$ matrix $\mathbf{A}$, then there must be some vector $\boldsymbol{v}$ in $\mathbb{R}^n$ for which the equation $(\mathbf{A} + 3\mathbf{I})\boldsymbol{x} = \boldsymbol{v}$ has no solution.

Short questions set 3. Consider

$$\mathbf{M}_a = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 2 & a \\ 1 & 1 & 4 & a^2 \\ 1 & -1 & 8 & a^3 \end{bmatrix}$$

(a) (0.5^{pts}) For which values of a is the determinant equal to zero?
(b) (0.5^{pts}) What is the determinant of the matrix $\mathbf{M}_a$?
(c) (0.5^{pts}) Find $\dim \mathcal{N}\left(\mathbf{A}\right)$ depending on the values of a.
(d) (0.5^{pts}) Consider $a = 0$ and solve the system $\left(\mathbf{M}_a\right)\boldsymbol{x} = \mathbf{0}$.

Based on MIT Course 18.06 Quiz 2, April 11, 2012

1.18. Final May 13/14

EXERCISE 1. Consider the matrix $\mathbf{A} = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & a \end{bmatrix}$.

(a) (0.5^{pts}) For which values for parameter a each of the following statements is true respectively?

1. $\mathbf{A}$ is invertible.
2. $\mathbf{A}$ is symmetric.
3. $\mathbf{A}$ is diagonalizable.

(b) (0.5^{pts}) For which values of a the matrix $\mathbf{A}$ is positive definite?
(c) (0.5^{pts}) For which values of a, zero ($\lambda = 0$) is an eigenvalue of $\mathbf{A}$? Find a corresponding eigenvector.

$\boxed{\text{For questions (d) and (e) consider } a = 3.}$

(d) (0.5^{pts}) Find an eigenvalue and an eigenvector for $\mathbf{A}^2$.
(e) (0.5^{pts}) Find the implicit equations of $\mathcal{C}(\mathbf{A})$. What is the dimension of $\mathcal{C}(\mathbf{A})$?

Variación de un ejercicio propuesto por Maria Jesus Moreta y Mercedes Vazquez

EXERCISE 2. Consider the linear system $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$, where $\mathbf{A} = \begin{bmatrix} 1 & 2 & 0 & m \\ 0 & 1 & -1 & 2 \\ 1 & 2 & 0 & 0 \\ 2 & 4 & 1 & 3 \end{bmatrix}$ and $\boldsymbol{b} = \begin{pmatrix} 2 \\ 2 \\ n \\ 2 \end{pmatrix}$.

(a) (0.5^{pts}) Find the rank of $\mathbf{A}$ depending on the values of m.
(b) (0.5^{pts}) Explain, depending on the values of m and n, when the system is not solvable, when it has infinitely many solutions, when it has a single unique solution? (the echelon form of augmented matrix could help you find the answer).
(c) (0.5^{pts}) Consider $m = 0$ and $n = 2$, and solve the system by gaussian elimination (if that is possible).
(d) (0.5^{pts}) Find (if it is possible) values of m such that the set of solutions to $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$ is a plane.
(e) (0.5^{pts}) Compute $\det(\mathbf{A})$ expanding along the first column of $\mathbf{A}$.

Variación de un ejercicio propuesto por Maria Jesus Moreta y Mercedes Vazquez

EXERCISE 3. Consider the linear system $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$, where

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 3 & 0 \\ -2 & 0 & 6 \end{bmatrix}; \quad \boldsymbol{b} = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}.$$

(a) (1^{pts}) Is system $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$ solvable? If it is solvable, find the solution.
(b) (0.5^{pts}) Complete the squares of the quadratic form $\boldsymbol{x}\mathbf{A}\boldsymbol{x}$. Is $\boldsymbol{x}\mathbf{A}\boldsymbol{x}$ positive definite?
(c) (0.5^{pts}) Is $\lambda = 0$ an eigenvalue of $\mathbf{A}$? Please, explain your answer.
(d) (0.5^{pts}) Is $\boldsymbol{v} = \begin{pmatrix} 0 \\ 4 \\ 0 \end{pmatrix}$ an eigenvector of $\mathbf{A}$? If the answer is affirmative, say which is the corresponding eigenvalue.

Ejercicio propuesto por Haydee Lugo

SHORT QUESTIONS SET 1. Find the implicit equations of the plane spanned by $(1, 1, 0, 1)$ and $(0, 0, 1, 1)$, that goes through $(0, 0, 0, 1)$.

SHORT QUESTIONS SET 2. True or false (to receive full credit you must explain your answers in a clear and concise way)

(a) If $\mathbf{A}$ is a n by n squared matrix and $\mathbf{A}^2 = \mathbf{I}$, then the rank of $\mathbf{A}$ is n.
(b) If $\mathbf{B}^2 = \mathbf{B}$ then $\mathbf{B} = \mathbf{I}$.
(c) If $\lambda = 0$ is an eigenvalue of $\mathbf{A}$ then the system $\mathbf{A}\boldsymbol{x} = \mathbf{0}$ has a single unique solution.

SHORT QUESTIONS SET 3. Find a basis for the following subspace:

$$\mathcal{W} = \left\{(x, y, z) \in \mathbb{R}^3 \text{ such that } 3x + 2y - z = 0 \text{ and } 2y + 4z = 0\right\}.$$

SHORT QUESTIONS SET 4. Consider the matrix $\mathbf{A} = \begin{bmatrix} x & 1/2 \\ 1/2 & y \end{bmatrix}$

(a) For which values of x and y matrix $\mathbf{A}$ is positive definite?
(b) For which values of x and y matrix $\mathbf{A}$ is orthonormal?.

SHORT QUESTIONS SET 5. True or false (to receive full credit you must explain your answers in a clear and concise way)

Consider a squared matrix $\mathbf{A}$ of order 2×2 such that $\det(\mathbf{A}) = -1$; then:

(a) $\det(\mathbf{A}^n) = (-1)^n$.
(b) Matrix $\mathbf{A}$ can't be idempotent.
(c) Matrix $\mathbf{A}$ is not definite.

1.19. Final July 12/13

EXERCISE 1. Consider the linear system, $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$ where $\mathbf{A} = \begin{bmatrix} 1 & 1 & 2 & 0 & -1 \\ 2 & 3 & 3 & -1 & a \\ 1 & 2 & 1 & -1 & 1 \end{bmatrix}$ and $\boldsymbol{b} = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}$.

(a) (0.5^{pts}) Find the echelon form of the augmented matrix.
(b) (0.5^{pts}) Describe the set of solutions to the system $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$ depending on the values of the parameter a.
(c) Consider the case $a = 1$.

1. (0.5^{pts}) How many variables can be consider as pivot (or dependent or endogenous) variables? Which ones?
2. (0.5^{pts}) Find the dimension and a basis of the set of solutions to $\mathbf{A}\boldsymbol{x} = \mathbf{0}$.
3. (0.5^{pts}) Find the set of solutions to $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$.

EXERCISE 2. Consider the matrices

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & b \\ 0 & -1 & -3 \\ 0 & 2 & 4 \end{bmatrix}; \quad \mathbf{B} = \begin{bmatrix} b & 1 & 3 & 5 \\ 1 & 8 & 2 & 3 \\ 3 & 2 & 4 & b \\ 5 & 3 & b & 0 \end{bmatrix}; \quad \mathbf{C} = \begin{bmatrix} 1 & 7 & 3 & b \\ 0 & 0 & 2 & 7 \\ 0 & 0 & -3 & b \\ 0 & 0 & 0 & 3 \end{bmatrix}.$$

(a) (1^{pts}) For each of these matrices, find the values of the parameter b that make the matrix diagonalizable.
(b) (0.5^{pts}) For which matrices will be possible to find an orthonormal basis of eigenvectors?
(c) (0.5^{pts}) Compute, when it is possible, the diagonal matrix associated with the matrix $\mathbf{A}^{-1}$ and a basis of eigenvectors.
(d) (0.5^{pts}) Find $\mathbf{A}^{-1}$ in the diagonalizable case.

EXERCISE 3. This question is about an m by n matrix $\mathbf{A}$ for which

$$\mathbf{A}\boldsymbol{x} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \text{ has } \textbf{no solutions} \quad \text{and} \quad \mathbf{A}\boldsymbol{x} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \text{ has } \textbf{exactly one solution}.$$

(a) (1^{pts}) Give all possible information about m and n and the rank r of $\mathbf{A}$.
(b) (1^{pts}) Find all solutions to $\mathbf{A}\boldsymbol{x} = \mathbf{0}$ and **explain your answer**.
(c) (0.5^{pts}) Write down an example of a matrix $\mathbf{A}$ that fits the description in part (a).

MIT Course 18.06 Quiz 1, Fall 2008

SHORT QUESTIONS SET 1.

(a) (0.5^{pts}) If we know that $\det \mathbf{A} = 5$, where $\mathbf{A} = \begin{bmatrix} 1 & 8 & 3 \\ x & y & z \\ -3 & 7 & 2 \end{bmatrix}$, what is the determinant of

$$\mathbf{B} = \begin{bmatrix} x & y & z \\ 1 & 8 & 3 \\ -3-4x & 7-4y & 2-4z \end{bmatrix}?$$

(b) (0.5^{pts}) Find an eigenvalue of $\mathbf{C} = \begin{bmatrix} 1 & 8 & 3 \\ u & v & w \\ u+1 & v+8 & w+3 \end{bmatrix}$.

SHORT QUESTIONS SET 2.

(a) (0.5^{pts}) Find the parametric equations of the plane passing through the points $\boldsymbol{a} = (1, 1, 0)$, $\boldsymbol{b} = (0, 0, 1)$ and $\boldsymbol{c} = (1, 1, 1)$
(b) (0.5^{pts}) Find a perpendicular vector to the plane in part (a).

SHORT QUESTIONS SET 3. Consider the following vectors: $\boldsymbol{u}_1 = (1, 1, 1)$, $\boldsymbol{u}_2 = (a, 1, 1)$ and $\boldsymbol{u}_3 = (1, c, 1)$.

(a) (0.5^{pts}) Find the values for a and b such that these vectors expand a 1-dimensional subspace.
(b) (0.5^{pts}) Find the values for a and b such that these vectors expand the full 3-dimensional subspace $\mathbb{R}^3$.

Short questions set 4. Consider a 2×2 matrix $\mathbf{A}$ with characteristic polynomial $p(\lambda) = \lambda^2 - 2\lambda$.

(a) (0.5^{pts}) Prove that the diagonal matrix $\mathbf{D}$, with the eigenvalues of $\mathbf{A}$ on its main diagonal, satisfies $\mathbf{D}^2 - 2\mathbf{D} = \mathbf{0}$.

(b) (0.5^{pts}) Prove that $\mathbf{A}^2 - 2\mathbf{A} = \mathbf{0}$.

Short questions set 5. Consider the matrix $\mathbf{A} = \begin{bmatrix} 1 & -1 & 1 \\ -1 & 3 & -1 \\ 1 & -1 & 1 \end{bmatrix}$.

(a) (0.5^{pts}) write the expression of the quadratic form $\boldsymbol{x}\mathbf{A}\boldsymbol{x}$ asociated to $\mathbf{A}$.

(b) (0.5^{pts}) Classify the above quadratic form (posititive, negative, definite, semi-definite, not definite. . . ?)

1.20. Final May 12/13

EXERCISE 1.

(a) (1^{pts}) Find the parametric equations of the plane Π :

$$\Pi : \qquad 3x - 5y + z + 3 = 0.$$

(b) (1^{pts}) Find the value of a so that the line r whose parametric equations are given by

$$r: \qquad \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -3 \end{pmatrix} + t \begin{pmatrix} 1 \\ -1 \\ a \end{pmatrix}$$

is in the plane Π.

(c) (0.5^{pts}) Find the implicit equations of the line r above.

EXERCISE 2.

(a) (1^{pts}) Consider the matrix $\mathbf{A} = \begin{bmatrix} 2 & 6 \\ a & b \end{bmatrix}$. Find the values a y b so that $\mathbf{A}$ has eigenvectors $\boldsymbol{x}_1 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$ and $\boldsymbol{x}_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$.

(b) (1^{pts}) Find a different matrix $\mathbf{B}$ with those same eigenvectors $\boldsymbol{x}_1 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$ and $\boldsymbol{x}_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$, and with eigenvalues $\lambda_1 = 1$ and $\lambda_2 = 0$. Compute $\mathbf{B}^{10}$.

(c) (0.5^{pts}) For wich numbers a is the matrix $\mathbf{C} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & a & 1 \end{bmatrix}$ diagonalizable?

EXERCISE 3. The following information is known about an $m \times n$ matrix $\mathbf{A}$:

$$\mathbf{A} \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}; \quad \mathbf{A} \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}; \quad \mathbf{A} \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 5 \\ 10 \end{pmatrix}.$$

(a) (0.5^{pts}) Is the set $\left\{ \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} \right\}$ a basis of $\mathbb{R}^3$?

(b) (1^{pts}) Give a matrix $\mathbf{C}$ and an invertible matrix $\mathbf{B}$ such that $\mathbf{A} = \mathbf{C}\mathbf{B}^{-1}$ (You *don't* have to evaluate $\mathbf{B}^{-1}$ or find $\mathbf{A}$ explicitly. Just say what $\mathbf{B}$ and $\mathbf{C}$ are and use them to reason about $\mathbf{A}$ in the subsequent parts).

(c) (0.5^{pts}) Find a basis for the left null space of $\mathbf{A}$; that is, a basis of $\mathcal{N}\left(\mathbf{A}^{\mathsf{T}}\right)$.

(d) (0.5^{pts}) What are m, n, and the rank r of $\mathbf{A}$?

SHORT QUESTIONS SET 1. Consider the matrix $\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 3 & 4 & 5 \\ 0 & 0 & 5 & 6 \\ 1 & 2 & 0 & 1 \end{bmatrix}$.

(a) (0.5^{pts}) Compute $\det \mathbf{A}$.

(b) (0.5^{pts}) What is the third component x_3 of the solution to $\mathbf{A} \begin{pmatrix} x_1 \\ \vdots \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}$?

SHORT QUESTIONS SET 2. Consider the following quadratic form

$$f(x, y, z) = x^2 + 3y^2 + 7z^2 - 2xy + 4xz - 8yz.$$

(a) (0.5^{pts}) Find the symmetric matrix $\mathbf{A}$ associated to $f(x, y, z)$.

(b) (0.5^{pts}) Prove $\mathbf{A}$ is positive definite.

SHORT QUESTIONS SET 3.

(a) (0.5$^{\text{pts}}$) If $\mathbf{A}$ and $\mathbf{B}$ are orthogonal matrices ($\mathbf{A}\mathbf{A}^{\mathsf{T}} = \mathbf{I}$ and $\mathbf{B}\mathbf{B}^{\mathsf{T}} = \mathbf{I}$), prove that the matrix $\mathbf{A}\mathbf{B}^{-1}$ is also orthogonal.

(b) (0.5$^{\text{pts}}$) Consider an $m \times n$ matrix $\mathbf{B}$ such that $\mathbf{B}^{\mathsf{T}}\mathbf{B}$ is invertible. Find the order of the matrix $\mathbf{C} = \mathbf{B}(\mathbf{B}^{\mathsf{T}}\mathbf{B})^{-1}\mathbf{B}^{\mathsf{T}}$ and prove that $\mathbf{C}^2 = \mathbf{C}$ (that is, prove that $\mathbf{C}$ is an idempotent matrix).

(c) (0.5$^{\text{pts}}$) Find a unit vector with the same direction as $\boldsymbol{v} = (2, -1, 0, 4, -2)$.

(d) (0.5$^{\text{pts}}$) Give an example of a 5×4 matrix with rank 3.

Short questions set 4. True or false (to receive full credit you must explain your answers in a clear and concise way)

(a) (0.5$^{\text{pts}}$) The set $B = \{(1, 0, 1), (1, 1, 0)\}$ is a basis of the subspace of solutions to $x - y - z = 0$.

(b) (0.5$^{\text{pts}}$) If a square matrix has a repeated eigenvalue, it cannot be diagonalizable.

MIT Course 18.06 Final Exam, December 13, 1993

1.21. Final September 11/12

EXERCISE 1. Consider the diagonalizable matrix $\mathbf{A}$. It is known that the following subspaces:

$$\mathcal{V}_1 = \{(x,y,z) \in \mathbb{R}^3 \text{ such as } y+z=0\} \quad \text{and} \quad \mathcal{V}_{1/2} = \{(x,y,z) \in \mathbb{R}^3 \text{ such as } x=0, y=0\}$$

are associated to the eigenvalues $\lambda = 1$ and $\lambda = \frac{1}{2}$ respectively. Find:

(a) (1^{pts}) The diagonal matrix $\mathbf{D}$ and the matrix $\mathbf{P}$ such as $\mathbf{A} = \mathbf{PDP}^{-1}$.
(b) (1^{pts}) The matrix $\mathbf{A}$.
(c) (0.5^{pts}) The matrix $\mathbf{M} = 2\mathbf{A}^4 - 7\mathbf{A}^3 + 9\mathbf{A}^2 - 5\mathbf{A} + \mathbf{I}$. (*Hint: please note that* $2(\frac{1}{2})^4 - 7(\frac{1}{2})^3 + 9(\frac{1}{2})^2 - 5(\frac{1}{2}) + 1 = 0$ *and also that* $2(1)^4 - 7(1)^3 + 9(1)^2 - 5(1) + 1 = 0$).

EXERCISE 2.

(a) (0.5^{pts}) $\mathbf{A}$ and $\mathbf{B}$ are any matrices with the same number of rows. What can you say (*and explain why it is true*) about the comparison of

$$\text{rank of } \mathbf{A} \qquad \text{and} \qquad \text{rank of the block matrix } \begin{bmatrix} \mathbf{A} & \mathbf{B} \end{bmatrix}.$$

(b) (1^{pts}) Suppose $\mathbf{B} = \mathbf{A}^2$. How do those ranks compare? Explain your reasoning.
(c) (1^{pts}) If $\mathbf{A}$ is m by n of rank r, what are the dimensions of these nullspaces?

$$\text{Nullspace of } \mathbf{A} \qquad \text{and} \qquad \text{Nullspace of } \begin{bmatrix} \mathbf{A} & \mathbf{A} \end{bmatrix}$$

MIT Course 18.06 Final, Fall 2006

EXERCISE 3. Consider the following system of linear equations $\begin{cases} x+y+z &= 3 \\ x-y+z &= 1 \\ 2x+az &= b \end{cases}$; where a and b are parameters

(a) (0.5^{pts}) Find, if it is possible, the values of a and b that makes the system solvable.
(b) (0.5^{pts}) Find, if it is possible, values of a and b such as the set of solutions is a plane.
(c) (1^{pts}) Find, if it is possible, values of a and b such as the set of solutions is a line. ¿Which variables can be choosen as free variables? Find, if it is possible, a basis for the set of solutions.
(d) (0.5^{pts}) Solve the system of equations when $a = 3$ and $b = 4$. Does the solution belong to the set of solutions of Part (c)? Explain you answer.

Problemas de Álgebra Lineal. Paloma Sanz, Francisco José Vázquez y Pedro Ortega. *Editorial: Pearson*

SHORT QUESTIONS SET 1.

(a) Find a parametric representation for the line passing through the points $(-1,2)$ y $(0,3)$.
(b) Find a implicit representation for the same line.

SHORT QUESTIONS SET 2. For wich numbers b does this matrix $\mathbf{C}$ have 3 positive eigenvalues?

$$\mathbf{C} = \begin{bmatrix} 2 & b & 3 \\ b & 2 & b \\ 3 & b & 4 \end{bmatrix}$$

MIT Course 18.06 Final Exam, May 16, 2005, and MIT Course 18.06 Quiz 3, December 5, 2005

SHORT QUESTIONS SET 3. Suppose $\mathbf{A}$ is a 5 by 3 matrix with orthonormal columns. Evaluate the following determinants:

(a) $\det \mathbf{A}^{\mathsf{T}}\mathbf{A}$
(b) $\det \mathbf{A}\mathbf{A}^{\mathsf{T}}$
(c) $\det \mathbf{A}(\mathbf{A}^{\mathsf{T}}\mathbf{A})^{-1}\mathbf{A}^{\mathsf{T}}$

MIT Course 18.06 Quiz 2, April 10, 1996

SHORT QUESTIONS SET 4. What value(s) of x give $\det \mathbf{A} = 0$, where

$$\mathbf{A} = \begin{bmatrix} x & 2 & 3 \\ -x & x & 0 \\ 3 & 2 & 5 \end{bmatrix}?$$

MIT Course 18.06 Quiz 2, April 10, 1996

SHORT QUESTIONS SET 5. Let $\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 1 & 3 \\ 4 & 2 & 3 & 1 \\ 3 & 1 & 4 & 2 \end{bmatrix}$. Show $\boldsymbol{v} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$ is an eigenvector for $\mathbf{A}$.

SHORT QUESTIONS SET 6. *True or false* (explain your answer):

Consider two squared and invertible matrices $\mathbf{A}$ and $\mathbf{B}$. Then

(a) $(\mathbf{A}^\mathsf{T}\mathbf{B}^\mathsf{T})^{-1} = (\mathbf{A}^{-1}\mathbf{B}^{-1})^\mathsf{T}$.

(b) If $\mathbf{A}$ and $\mathbf{B}$ are both also orthonormal, then, $\mathbf{AB}$ is orthonormal.

1.22. Final June 11/12

EXERCISE 1. Consider the matrix $\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 2 \\ 0 & 1 & 0 & 1 \end{bmatrix}$.

(a) (1^{pts}) Compute the echelon form of $\mathbf{A}$ by gaussian elimination. Are its columns linearly independent? Compute the rank of $\mathbf{A}$.
(b) (1^{pts}) Describe the sub-space spanned by the three first columns. What is its dimension? Is it different if we include the fourth column?
(c) (0.5^{pts}) Solve the system $\mathbf{A}\boldsymbol{x} = \mathbf{0}$. How many variables can be chosen as free (or exogenous) variables? Which ones? What is the dimension of $\mathcal{N}(\mathbf{A})$?

Propuesto por Mercedes Vazquez.

EXERCISE 2. Consider the following matrix

$$\mathbf{A} = \begin{bmatrix} a & 0 & b \\ 0 & 0 & 1 \\ b & 1 & 0 \end{bmatrix}$$

(a) (0.5^{pts}) Study in which cases (depending on a and b) are $\mathbf{A}$ and $\mathbf{A}^{-1}$ diagonalizable matrices.
(b) Consider $a = 1$ and $b = 0$.

1. (1^{pts}) Diagonalize $\mathbf{A}$ using an orthonormal basis of $\mathbb{R}^3$
2. (0.5^{pts}) Find a basis in $\mathbb{R}^3$ that allows you to diagonalize $\mathbf{A}^{-1}$; and write the associate diagonal matrix.
3. (0.5^{pts}) Prove $\boldsymbol{u} = (0, 2, 2)$ is an eigenvector of $\mathbf{A}$, and compute $\mathbf{A}^{10}\boldsymbol{u}$.

Propuesto por Rafael Lopez Zorzano.

EXERCISE 3. (You don't need too much computing here!)

(a) (0.5^{pts}) Are the vectors $\boldsymbol{x}_1 = \begin{pmatrix} -2 \\ -1 \\ 3 \\ 4 \end{pmatrix}$ and $\boldsymbol{x}_2 = \begin{pmatrix} -8 \\ 2 \\ -2 \\ 1 \end{pmatrix}$ linearly independent? Are these vectors perpendicular to each other? Explain.
(b) (0.5^{pts}) Are the vectors $\boldsymbol{x}_1 = \begin{pmatrix} -2 \\ -1 \\ 3 \\ 4 \end{pmatrix}$, $\boldsymbol{x}_2 = \begin{pmatrix} -8 \\ 2 \\ -2 \\ 1 \end{pmatrix}$, $\boldsymbol{x}_3 = \begin{pmatrix} 10 \\ 1 \\ 1 \\ 6 \end{pmatrix}$, $\boldsymbol{x}_4 = \begin{pmatrix} -2 \\ -1 \\ 3 \\ 4 \end{pmatrix}$, a basis for $\mathbb{R}^4$? Explain.
(c) (0.5^{pts}) Are the vectors $\boldsymbol{x}_1 = \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix}$, $\boldsymbol{x}_2 = \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$, $\boldsymbol{x}_3 = \begin{pmatrix} -4 \\ -2 \\ 2 \\ 1 \end{pmatrix}$, solution to the system $x_1 + 2x_2 + 3x_3 + 6x_4 = 0$? Are this vectors a basis for the 3-dimensional subspace described by this homogeneous system? Explain.
(d) (1^{pts}) Find the value for q for which the vectors $\begin{pmatrix} 1 \\ 4 \\ 6 \end{pmatrix}$, $\begin{pmatrix} 0 \\ 2 \\ 2 \end{pmatrix}$, $\begin{pmatrix} -1 \\ 12 \\ 10 \end{pmatrix}$, $\begin{pmatrix} q \\ 3 \\ 1 \end{pmatrix}$, do not span $\mathbb{R}^3$.

MIT Course 18.06 March, 1996

SHORT QUESTIONS SET 1.

(a) Find the determinant of $\mathbf{A}$ and the determinant of $\mathbf{A}^{-1}$ if

$$\mathbf{A} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 3 \\ 1 & 3 & 1 & 7 \end{bmatrix}$$

(b) Find the $(1, 2)$ entry of $\mathbf{A}^{-1}$

MIT Course 18.06 Quiz 1, April 1, 2005

SHORT QUESTIONS SET 2. Suppose the following information is known about $\mathbf{A}$:

$$\mathbf{A}\begin{pmatrix}1\\2\\1\end{pmatrix}=6\begin{pmatrix}1\\2\\1\end{pmatrix},\quad \mathbf{A}\begin{pmatrix}1\\-1\\1\end{pmatrix}=3\begin{pmatrix}1\\-1\\1\end{pmatrix},\quad \mathbf{A}\begin{pmatrix}1\\0\\-1\end{pmatrix}=3\begin{pmatrix}1\\0\\-1\end{pmatrix},$$

In each of these questions, you must give a correct reason to get full credit.

(a) Find the eigenvalues and the eigenvectors of $\mathbf{A}$
(b) Is $\mathbf{A}$ a diagonalizable matrix? Is $\mathbf{A}$ an invertible matrix?
(c) What are the trace and determinant of $\mathbf{A}$?
(d) Is $\mathbf{A}$ a symmetric matrix?

Based on MIT Course 18.06 Quiz 3, May 8, 1996

SHORT QUESTIONS SET 3. Consider the following quadratic form

$$f(\boldsymbol{x}) = x_1^2 + 2a\,x_1x_2 + 2a\,x_2x_3 + x_3^2,$$

compute (if it is possible) all the values for a such as $f(\boldsymbol{x})$ is negative definite.
Propuesto por el profesor Rafael A. Lopez Zorzano

SHORT QUESTIONS SET 4. True or false (to receive full credit you must explain your answers in a clear and concise way)

(a) If $\lambda = 0$ is an eigenvalue of the n by n matrix $\mathbf{A}$, then $\operatorname{rg}\big(\mathbf{A}\big) < n$.
(b) If $\lambda = -3$ is an eigenvalue of $\underset{n\times n}{\mathbf{A}}$, then there exists a vector $\boldsymbol{b} \in \mathbb{R}^n$ such as the system $(\mathbf{A}+3\mathbf{I})\boldsymbol{x} = \boldsymbol{b}$ is not solvable.
(c) If $\lambda = 0$ is an eigenvalue of $\mathbf{A}$ then the system $\mathbf{A}\boldsymbol{x} = \mathbf{0}$ has a sole solution.

1.23. Final September 10/11

EXERCISE 1. Consider the following matrix

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 1 & a & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

(a) (0.5^{pts}) Prove $\mathbf{A}$ is invertible if and only if $a \neq 0$.
(b) (0.5^{pts}) Is $\mathbf{A}$ positive definite when $a = 1$? Explain your answer.
(c) (1^{pts}) Compute $\mathbf{A}^{-1}$ when $a = 2$.
(d) (0.5^{pts}) How many variables can be chosen as pivot (or exogenous) variables in the system $\mathbf{A}\boldsymbol{x} = \boldsymbol{o}$ when $a = 0$? Which ones?

EXERCISE 2. Let $\mathbf{A}$ be the matrix

$$\mathbf{A} = \begin{pmatrix} 4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 1 & 0 & 0 & 2 \end{pmatrix}$$

(a) (1^{pts}) Determine if $\mathbf{A}$ is diagonalizable, and if so, diagonalize it.
(b) (0.5^{pts}) Compute $\left(\mathbf{A}^6\right)\boldsymbol{v}$, where $\boldsymbol{v} = \begin{pmatrix} 0 & 0 & 0 & 1 \end{pmatrix}$.
(c) (0.5^{pts}) Using the the eigenvalues found in part (a) justify that $\mathbf{A}$ is invertible.
(d) (0.5^{pts}) What is the relation between the eigenvalues of $\mathbf{A}$ and the eigenvalues of $\mathbf{A}^{-1}$?

EXERCISE 3. Consider the system $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$, where

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 0 & 1 & 1 \\ 0 & 0 & 2 & 3 & 1 \\ 0 & 0 & 1 & 4 & 2 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}, \qquad \boldsymbol{b} = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 2 \end{pmatrix}.$$

(a) (1^{pts}) Find the solution to the system.
(b) (0.5^{pts}) Explain why the solution set is a line in $\mathbb{R}^5$. Find a direction vector (a vector parallel to the line) and any point on that line.
(c) (1^{pts}) Find the set of vectors perpendicular to the solution set. Prove that set is a four dimensional subspace. Find a basis for that subspace.

SHORT QUESTIONS SET 1. True or false (to receive full credit you must explain your answers in a clear and concise way)

(a) If $\mathbf{A}$ is symetric, then so it is $\mathbf{A}^2$.
(b) If $\mathbf{A}^2 = \mathbf{A}$ then $(\mathbf{I} - \mathbf{A})^2 = (\mathbf{I} - \mathbf{A})$ where $\mathbf{I}$ is the identity matrix.
(c) If $\lambda = 0$ is an eigenvalue of the squared matrix $\mathbf{A}$, then the linear system $\mathbf{A}\boldsymbol{x} = \mathbf{0}$ is is always solvable and has only one solution.
(d) If $\lambda = 0$ is an eigenvalue of the squared matrix $\mathbf{A}$, then the linear system $\mathbf{A}\boldsymbol{x} = \mathbf{0}$ could be unsolvable.
(e) If a matrix is orthogonal (perpendicular columns of norm one), then so it is the inverse of that matrix.
(f) If 1 is the only eigenvalue of a 2×2 matrix $\mathbf{A}$, then $\mathbf{A}$ must be the identity matrix $\mathbf{I}$.

SHORT QUESTIONS SET 2. The matrix

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 1 & -7 & 1 \\ 2 & 4 & 1 & -5 & 0 \\ 1 & 2 & 2 & -16 & 3 \end{bmatrix}$$

$\mathbf{A}$ is converted to row-reduced echelon form by the usual row-elimination steps, resulting in the matrix:

$$\mathbf{R} = \begin{bmatrix} 1 & 2 & 0 & 2 & 1 \\ 0 & 0 & 1 & -9 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(a) What is the minimum number of columns of $\mathbf{A}$ that form a dependent set of vectors.
(b) What is the maximum number of columns of $\mathbf{A}$ that forms an *in*dependent set of vectors.

Versión del ejercicio: MIT Course 18.06 Quiz 2. Spring, 2009

Short questions set 3.

(a) Consider the quadratic form $q(x, y, z) = x^2 + 2xy + ay^2 + 8z^2$ and find its corresponding symmetric matrix $\mathbf{Q}$; determine if $\mathbf{Q}$ is positive-definite, positive-semidefinite, negative-definite, negative-semidefinite or indefinite when the parameter a is equal to one ($a = 1$).

(b) If $a \neq 1$, determine whether the matrix is positive-definite, positive-semidefinite, negative-definite, negative-semidefinite or indefinite.

1.24. Final June 10/11

EXERCISE 1. Consider the following matrices

$$\mathbf{A}=\begin{bmatrix}2&1&0&0\\1&2&0&0\\0&0&a&0\\0&0&a&a\end{bmatrix}\qquad\text{and}\qquad\mathbf{B}=\begin{bmatrix}2&1&0\\1&2&0\\0&0&1\end{bmatrix}.$$

(a) (0.5^{pts}) Compute the eigenvalues of $\mathbf{A}$.
(b) (0.5^{pts}) Prove that when $a=2$ the matrix $\mathbf{A}$ is not diagonalisable.
(c) (1^{pts}) For matrix $\mathbf{B}$, find a diagonal matrix $\mathbf{D}$ and an orthonormal matrix $\mathbf{P}$ such as $\mathbf{B}=\mathbf{PDP}^{\intercal}$.
(d) (0.5^{pts}) Find the quadratic form $f(x,y,z)$ associated to $\mathbf{B}$, and prove it is positive defined.

Versión de un ejercicio proporcionado por Mercedes Vazquez

EXERCISE 2. Consider the matrices

$$\mathbf{A}=\begin{bmatrix}1&4&2&3\\2&3&3&7\\0&1&0&3\\0&2&0&a\end{bmatrix}\qquad\text{and}\qquad\mathbf{B}=\begin{bmatrix}0&1&1&0\\1&0&0&1\\0&0&0&1\\0&1&0&0\end{bmatrix};\qquad\text{and the vector}\qquad\boldsymbol{b}=\begin{pmatrix}1\\0\\0\\0\end{pmatrix}.$$

(a) (0.5^{pts}) For wich values of a the matrix $\mathbf{A}$ is invertible?
(b) (1^{pts}) Consider $a=5$. Using the Cramer's rule, compute the fourth coordinate x_4 of $\boldsymbol{x}$ for linear system $\mathbf{A}\boldsymbol{x}=\boldsymbol{b}$.
(c) (1^{pts}) Compute $\mathbf{B}^{-1}$. Use the matrix $\mathbf{B}^{-1}$ to solve $\mathbf{B}\boldsymbol{x}=\boldsymbol{b}$.

Proporcionado por Mercedes Vazquez

EXERCISE 3. Consider the linear system

$$\begin{bmatrix}2&1&2\\4&1&2\\2&1&a\end{bmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}.$$

(a) (0.5^{pts}) Find the echelon form of the coefficient matrix. For which values of a the system has one and only one solution.
(b) (1^{pts}) Consider $a=2$. How many variables are free? which ones can be chosen as free variables? (*remember that column exchange is possible!*). Find the dimension of the null space of the coefficient matrix $\mathbf{A}$, find a basis of $\mathcal{N}\left(\mathbf{A}\right)$, and solve the system.
(c) (1^{pts}) Consider the non-linear system

$$\begin{cases}x^2+\frac{y^2}{2}+4\sqrt{z} &=5.5\\ 2x^2+y+2z &=5\end{cases}$$

Is the vector (1,1,1) a solution to the system? Compute an approximate solution when $z=1.1$.

Versión de un ejercicio proporcionado por Mercedes Vazquez

EXERCISE 4. Consider the linear system

$$\mathbf{A}\boldsymbol{x}=\begin{pmatrix}2\\4\\2\end{pmatrix};\qquad\text{with solution }\boldsymbol{x}=\begin{pmatrix}2\\0\\0\end{pmatrix}+c\begin{pmatrix}1\\1\\0\end{pmatrix}+d\begin{pmatrix}0\\0\\1\end{pmatrix}$$

(a) (1^{pts}) Find the dimension of the row space of $\mathbf{A}$. Explain your answer.
(b) (1^{pts}) Construct the matrix $\mathbf{A}$. Explain your answer.
(c) (0.5^{pts}) For which right hand side vectors $\boldsymbol{b}$ the system $\mathbf{A}\boldsymbol{x}=\boldsymbol{b}$ is solvable?

SHORT QUESTIONS SET 1. Consider matrices $\underset{3\times3}{\mathbf{A}}$ and $\underset{3\times3}{\mathbf{B}}$ such as $\det(\mathbf{A})=2$ and $\det(\mathbf{B})=-2$

(a) (0.5^{pts}) Compute the determinants of $\mathbf{AB}^2$ and $(\mathbf{AB})^{-1}$
(b) (0.5^{pts}) Is it possible to compute the rank of $\mathbf{A}+\mathbf{B}$? and the rank of $\mathbf{AB}$?

SHORT QUESTIONS SET 2. Given the matrix $\mathbf{A} = \left(\begin{smallmatrix} a & 3/5 \\ b & 4/5 \end{smallmatrix}\right)$, compute the values (if they exist) of a and b such as

(a) (0.5^{pts}) $\mathbf{A}$ is ortho-normal.
(b) (0.5^{pts}) Columns of $\mathbf{A}$ are linearly independent.
(c) (0.5^{pts}) $\lambda = 0$ is an eigenvalue of $\mathbf{A}$.
(d) (0.5^{pts}) $\mathbf{A}$ is a symmetric definite negative matrix.

SHORT QUESTIONS SET 3. Consider the following linear system $\mathbf{A}\boldsymbol{x} = \mathbf{0}$ where

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 1 & 2 & 3 \\ a & 1 & 1 & 2 \end{bmatrix}.$$

(a) (0.5^{pts}) Find the values of a such as the set of solutions of the linear system is a line.
(b) (0.5^{pts}) Find the values of a such as the set of solutions of the linear system is a plane?

SHORT QUESTIONS SET 4.

(a) (0.5^{pts}) Find an homogeneous system $\mathbf{A}\boldsymbol{x} = \mathbf{0}$ such as its solutions set is

$$\left\{ \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = \alpha \begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \end{pmatrix} + \beta \begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \end{pmatrix} + \gamma \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}; \qquad \alpha, \beta, \gamma \in \mathbb{R}. \right\}$$

(b) (0.5^{pts}) If the characteristic polynomial of a matrix $\mathbf{A}$ is $p(\lambda) = \lambda^5 + 3\lambda^4 - 24\lambda^3 + 28\lambda^2 - 3\lambda + 10$, find the rank of $\mathbf{A}$.

1.25. Final September 09/10

EXERCISE 1. Consider the matrix

$$\mathbf{A} = \begin{bmatrix} a & 1 & 1 \\ 0 & 3 & 1 \\ 0 & 0 & 2 \end{bmatrix}.$$

(a) (0.5 pts) Prove that $\mathbf{A}$ is not diagonalizable when $a = 3$.
(b) (1 pt) Is $\mathbf{A}$ diagonalizable when $a = 2$? (explain). If it is diagonalizable, find an eigenvalue diagonal matrix $\mathbf{D}$ and an eigenvector matrix $\mathbf{S}$ such as $\mathbf{A} = \mathbf{SDS}^{-1}$.
(c) Is $\mathbf{A}^\mathsf{T}\mathbf{A}$ diagonalizable for any value a? Is it possible to find a full set of orthonormal eigenvectors of $\mathbf{A}^\mathsf{T}\mathbf{A}$?
(d) Find all posible values a such as $\mathbf{A}$ is invertible and diagonalizable.

EXERCISE 2. Consider the following system of linear equations

$$\begin{cases} x_1 + 2x_2 - x_3 + x_4 = -1 \\ -x_1 - 2x_2 + 3x_3 + 5x_4 = -5 \\ -x_1 - 2x_2 - x_3 - 7x_4 = 7 \end{cases}$$

(a) (0.5 pts) What is the rank of the coeficient matrix?
(b) (1.5 pts) Find all solutions to the system of linear equations
(c) (0.5 pts) Describe the geometric shape of the collection of all solutions to the above equations considered as a subset of $\mathbb{R}^4$.

EXERCISE 3. We have a 3×3 matrix $\mathbf{A} = \begin{bmatrix} a & 1 & 2 \\ b & 3 & 4 \\ c & 5 & 6 \end{bmatrix}$ with $\det \mathbf{A} = 3$. Compute the determinant of the following matrices:

(a) (0.5 pts)

$$\begin{bmatrix} a-2 & 1 & 2 \\ b-4 & 3 & 4 \\ c-6 & 5 & 6 \end{bmatrix}$$

(b) (0.5 pts)

$$\begin{bmatrix} 7a & 7 & 14 \\ b & 3 & 4 \\ c & 5 & 6 \end{bmatrix}$$

(c) (1 pts)

$$(2\mathbf{A})^{-1}\mathbf{A}^\mathsf{T}$$

(d) (0.5 pts)

$$\begin{bmatrix} a-2 & 1 & 2 \\ b & 3 & 4 \\ c & 5 & 6 \end{bmatrix}$$

SHORT QUESTIONS SET 1. Consider a linear system of algebraic equation $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$. Here the matrix A has three rows and four columns.

(a) Does such a linear system always have at least one solution? If not provide an example for which no solution exists.
(b) Can such a linear system have a unique solution? If so, provide and example of a problem with this property.
(c) Formulate, if possible, necessary and sufficient conditions on $\mathbf{A}$ and $\boldsymbol{b}$ which guarantee that at least one solution exists.
(d) Formulate, if possible, necessary and sufficient conditions on $\mathbf{A}$ which guarantee that at least one solution exists for any choice of $\boldsymbol{b}$.

SHORT QUESTIONS SET 2.

(a) Consider the matrix

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix}$$

Find $\mathbf{A}^{-1}$.

(b) Find a unit vector with the same direction as $\boldsymbol{v} = (2, -1, 0, 4, -2)$.

(c) Consider the following quadratic form

$$q(x, y, z) = x^2 + 6xy + y^2 + az^2;$$

Decide for which values a the quadratic form is positive definite, negative definite, semidefinite, or indefinite.

(d) Compute the following determinant:

$$\begin{vmatrix} 0 & 0 & 0 & 3 & 0 \\ -2 & 0 & 0 & 2 & 0 \\ 8 & -1 & 0 & -7 & 2 \\ -1 & 2 & 2 & 3 & 2 \\ 2 & 2 & 3 & 6 & 4 \end{vmatrix}$$

(e) Let $\mathbf{A}$ be a 2×2 matrix such that $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$ is an eigenvector for $\mathbf{A}$ with eigenvalue 2, and $\begin{pmatrix} 2 \\ -1 \end{pmatrix}$ is another eigenvector for $\mathbf{A}$ with eigenvalue -2. If $\boldsymbol{v} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$, compute $\left(\mathbf{A}^3\right)\boldsymbol{v}$.

(f) True or false? (to receive full credit you must explain your answers in a clear and concise way)

If $\mathbf{A}^{\mathsf{T}} = 2\mathbf{A}$, then the rows of $\mathbf{A}$ are linearly dependent.

1.26. Final June 09/10

Exercise 1. By performing row eliminations on the following 4×7 matrix $\mathbf{A}$

$$\mathbf{A} = \begin{bmatrix} 1 & -1 & 0 & 2 & 0 & -1 & 0 \\ 2 & -2 & 1 & 5 & 0 & -1 & 0 \\ -3 & 3 & -1 & -7 & 2 & 2 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 1 \end{bmatrix}$$

we got the following matrix $\mathbf{B}$:

$$\mathbf{B} = \begin{bmatrix} 1 & -1 & 0 & 2 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

(a) What is the rank of $\mathbf{A}$?
(b) Find the complete solution to $\mathbf{A}\boldsymbol{x} = \mathbf{0}$.
(c) Write, if it is possible, the general solution for $\mathbf{A}\boldsymbol{x} = \mathbf{0}$ as a function of x_2, x_4, and x_6.
(d) Is it possible to find a vector $\boldsymbol{b}$ in $\mathbb{R}^4$ that is not in the column space of $\mathbf{A}$ ($\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$ has no solution)? If it is, give an example.
(e) Give a right hand side vector $\boldsymbol{b}$ such as the vector $\boldsymbol{x} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$ is a solution to the system $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$.

versión modificada de un ejercicio de MIT Course 18.06 Quiz 1, October 4, 2004

Exercise 2. Consider the matrix

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix}$$

(a) Find the eigenvalues and eigenvectors of $\mathbf{A}$.
(b) Is $\mathbf{A}$ diagonalizable?
(c) Is it possible to find a matrix $\mathbf{P}$ such as $\mathbf{A} = \mathbf{P}\mathbf{D}\mathbf{P}^{\mathsf{T}}$, where $\mathbf{D}$ is diagonal?
(d) Find $|\mathbf{A}^{-1}|$.

Exercise 3. Consider the following system of linear equations

$$\begin{cases} x - y + 2z & = 1 \\ 2x - 3y + mz & = 3 \\ -x + 2y + 3z & = 2m \end{cases}$$

(a) Show that the system has solution for any value m
(b) Find the solution when $m = -1$.
(c) Is the set of solutions to the system in the last question ($m = -1$) a line in $\mathbb{R}^3$? Is there any m such as the set of solutions to the system is a plane in $\mathbb{R}^3$?... and a point in $\mathbb{R}^3$?
(d) Find the solution to the system when $m = 1$.

Short questions set 1. Consider the squared matrices $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$. True or false? (to receive full credit you must explain your answers in a clear and concise way)

(a) If $\mathbf{AB} = \mathbf{I}$ and $\mathbf{CA} = \mathbf{I}$ then $\mathbf{B} = \mathbf{C}$.
(b) $(\mathbf{AB})^2 = \mathbf{A}^2\mathbf{B}^2$.
(c) $|\mathbf{A}\mathbf{A}^{\mathsf{T}}| = |\mathbf{A}|^2$.

Short questions set 2. Consider a 3 by 3 matrix $\mathbf{A}$ with eigenvalues $\lambda_1 = 1$, $\lambda_2 = 2$, and $\lambda_3 = -1$; and let $\boldsymbol{v}_1 = (1, 0, 1)$ and $\boldsymbol{v}_2 = (1, 1, 1)$ be the corresponding eigenvectors to λ_1 and λ_2.

(a) Is $\mathbf{A}$ diagonalizable?
(b) Is $\boldsymbol{v}_3 = (-1, 0, -1)$ an eigenvector associated to the eigenvalue $\lambda_3 = -1$?
(c) Compute $\mathbf{A}(\boldsymbol{v}_1 - \boldsymbol{v}_2)$.

Short questions set 3. Consider the matrix

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & a & 1 \\ 1 & a & 0 & 0 \\ 0 & 0 & 1 & a \\ 2 & 2a & 0 & 1 \end{bmatrix}$$

(a) Prove that $\mathbf{A}$ is invertible for any value of a.
(b) Compute $\mathbf{A}^{-1}$ when $a = 0$.

SHORT QUESTIONS SET 4. Consider the following quadratic forms

$$\begin{aligned} q_1(x, y, z) &= x^2 + 4y^2 + 5z^2 - 4xy. \\ q_2(x, y, z) &= -x^2 + 4y^2 + z^2 + 2xy - 2axz. \end{aligned}$$

(a) Show that $q_1(x, y, z)$ is positive semi-definite.
(b) Find, if it is possible, any value of a such as $q_2(x, y, z)$ is negative definite.

References

Strang, G. (). 18.06 linear algebra. Massachusetts Institute of Technology: MIT OpenCourseWare. License: Creative Commons BY-NC-SA.
URL http://ocw.mit.edu

Soluciones

(Final June 22/23) Exercise 1(a) Since $\boldsymbol{u} \in \mathcal{W}$, the vectors of $\mathcal{W}$ belong to $\mathbb{R}^3$, therefore, $\boxed{n=3}$. Since the dimension of $\mathcal{W} = \mathcal{N}\left(\mathbf{A}\right)$ is 1, the $\boxed{\text{range of } \mathbf{A} \text{ is } 2}$. It is impossible to know the number of rows of $\mathbf{A}$, we can only say that $\boxed{m \geq 2}$ (otherwise the range could not be 2).

□

(Final June 22/23) Exercise 1(b) The solution set is $\left\{ \boldsymbol{v} \in \mathbb{R}^3 \;\middle|\; \exists \boldsymbol{p} \in \mathbb{R}^1,\ \boldsymbol{v} = \begin{pmatrix}1\\0\\0\end{pmatrix} + \begin{bmatrix}1\\1\\0\end{bmatrix}\boldsymbol{p} \right\}$, since $\mathbf{A}\begin{pmatrix}1\\0\\0\end{pmatrix} = \mathbf{A}_{|1}$ and $\left[\begin{pmatrix}1\\1\\0\end{pmatrix};\right]$ is a basis for $\mathcal{N}\left(\mathbf{A}\right) = \mathcal{W}$.

□

(Final June 22/23) Exercise 1(c) It is enough that its rows (of $\mathbb{R}^3$) are perpendicular to $\boldsymbol{u}$. For example $\mathbf{A} = \begin{bmatrix}1 & -1 & 0\\0 & 0 & 1\end{bmatrix}$.

□

(Final June 22/23) Exercise 1(d) Given that $\mathbf{BA}\boldsymbol{x} = \mathbf{0} \;\Rightarrow\; \mathbf{B}^{-1}\mathbf{BA} = \mathbf{B}^{-1}\mathbf{0} = \mathbf{0}$ we have $\mathcal{N}\left(\mathbf{BA}\right) \subset \mathcal{N}\left(\mathbf{A}\right)$. And since $\mathbf{A}\boldsymbol{x} = \mathbf{0} \Rightarrow \mathbf{BA} = \mathbf{B0} = \mathbf{0}$, we have $\mathcal{N}\left(\mathbf{A}\right) \subset \mathcal{N}\left(\mathbf{BA}\right)$.
Hence, $\mathcal{N}\left(\mathbf{A}\right) = \mathcal{N}\left(\mathbf{B}\right)$, so $\mathcal{W} = \mathcal{V}$. Therefore $\mathcal{V} = \mathcal{W} = \left\{ \boldsymbol{v} \in \mathbb{R}^3 \;\middle|\; \exists \boldsymbol{p} \in \mathbb{R}^1,\ \boldsymbol{v} = \begin{bmatrix}1\\1\\0\end{bmatrix}\boldsymbol{p} \right\}$.

□

(Final June 22/23) Exercise 1(e) Dado que el rango es 2, la dimensión de su espacio fila es 2. Dado que las filas deben ser perpendicules a $\boldsymbol{u}$, unas ecuaciones cartesianas son $\mathcal{C}\left(\mathbf{A}^{\intercal}\right) = \left\{ \boldsymbol{v} \in \mathbb{R}^3 \;\middle|\; \begin{bmatrix}1 & 1 & 0\end{bmatrix}\boldsymbol{v} = (0,) \right\}$.

□

(Final June 22/23) Exercise 2(a) Using the first clue:

$$n = \operatorname{rg}(\mathbf{C}^{\intercal}\mathbf{C}) \leq \operatorname{rg}(\mathbf{C}) \leq n \quad \Rightarrow \quad \operatorname{rg}(\mathbf{C}) = n.$$

Using the second clue: *Since the columns of* $\mathbf{C}^{\intercal}\mathbf{C}$ *are a linear combination of those of* $\mathbf{C}^{\intercal}$:

$$\mathcal{C}\left(\mathbf{C}^{\intercal}\mathbf{C}\right) \subset \mathcal{C}\left(\mathbf{C}^{\intercal}\right) \quad \Rightarrow \quad n = \dim \mathcal{C}\left(\mathbf{C}^{\intercal}\mathbf{C}\right) \leq \dim \mathcal{C}\left(\mathbf{C}^{\intercal}\right) \leq n \quad \Rightarrow \quad \dim \mathcal{C}\left(\mathbf{C}^{\intercal}\right) = \operatorname{rg}(\mathbf{C}) = n.$$

□

(Final June 22/23) Exercise 2(b)

$$\boldsymbol{w} = \boldsymbol{u} \quad \Rightarrow \quad (\mathbf{C}^{\intercal}\mathbf{C})^{-1}(\mathbf{C}^{\intercal})\boldsymbol{z} = \mathbf{0} \quad \Rightarrow \quad (\mathbf{C}^{\intercal})\boldsymbol{z} = (\mathbf{C}^{\intercal}\mathbf{C})\mathbf{0} = \mathbf{0} \quad \Rightarrow \quad \mathbf{C}_{|j} \perp \boldsymbol{z} \quad \text{for } j = 1:n.$$

□

(Final June 22/23) Exercise 2(c) Since $\mathbf{C}$ is orthogonal, $\mathbf{C}(\mathbf{C}^{\intercal}) = \mathbf{I} = (\mathbf{C}^{\intercal})\mathbf{C}$; therefore

$$\|\boldsymbol{w} - \boldsymbol{u}\|^2 \;=\; \|(\mathbf{C}^{\intercal})\boldsymbol{z}\|^2 \;=\; (\mathbf{C}^{\intercal})\boldsymbol{z} \cdot (\mathbf{C}^{\intercal})\boldsymbol{z} \;=\; \boldsymbol{z}(\mathbf{C})(\mathbf{C}^{\intercal})\boldsymbol{z} \;=\; \boldsymbol{z} \cdot \boldsymbol{z} \;=\; \|\boldsymbol{z}\|^2.$$

□

(Final June 22/23) Exercise 2(d) It is diagonalizable because *it is symmetric*, since $\left((\mathbf{C}^{\intercal})\mathbf{C}\right)^{\intercal} = (\mathbf{C}^{\intercal})\left((\mathbf{C}^{\intercal})^{\intercal}\right) = (\mathbf{C}^{\intercal})\mathbf{C}$.

□

(Final June 22/23) Exercise 2(e)

$$\left[\begin{array}{ccc}1 & 1 & 0\\1 & 2 & 0\\0 & 0 & 1\\\hline 1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{array}\right] \xrightarrow[{[(-1)\mathbf{1}+\mathbf{2}]}]{\boldsymbol{\tau}} \left[\begin{array}{ccc}1 & 0 & 0\\1 & 1 & 0\\0 & 0 & 1\\\hline 1 & -1 & 0\\0 & 1 & 0\\0 & 0 & 1\end{array}\right] \xrightarrow[{[(-1)\mathbf{2}+\mathbf{1}]}]{\boldsymbol{\tau}} \left[\begin{array}{ccc}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\\\hline 2 & -1 & 0\\-1 & 1 & 0\\0 & 0 & 1\end{array}\right] \quad \Rightarrow \quad \left((\mathbf{C}^{\intercal})\mathbf{C}\right)^{-1} = \begin{bmatrix}2 & -1 & 0\\-1 & 1 & 0\\0 & 0 & 1\end{bmatrix}.$$

□

(Final June 22/23) Exercise 3(a) Since $\mathbf{D}$ is diagonal, each of its columns has the following form: $\mathbf{D}_{|j} = d_{jj}(\mathbf{I}_{|j})$. So, multiplying by $\mathbf{P}$ on both sides of $\mathbf{A} = \mathbf{P}\mathbf{D}\mathbf{P}^{-1}$ we get $\mathbf{AP} = \mathbf{PD}$. Therefore

$$\mathbf{AP}_{|j} = \mathbf{PD}_{|j} = d_{jj}\mathbf{P}(\mathbf{I}_{|j}) = (d_{jj})\mathbf{P}_{|j};$$

that is, when multiplying $\mathbf{A}$ by each column of $\mathbf{P}$ we obtain a multiple of that column.

□

(Final June 22/23) Exercise 3(b) Since the first two columns of $\mathbf{P}$ are eigenvectors corresponding to the eigenvalue $\lambda = 1$, it is enough to change one of them to a linear combination of both, so that the first two columns are no longer perpendicular to each other. For example, adding the second column to the first $\mathbf{C} = \Big[\mathbf{P}_{|1};\ (\mathbf{P}_{|2} + \mathbf{P}_{|1});\ \mathbf{P}_{|3};\Big] = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & -2 \end{bmatrix}$.

□

(Final June 22/23) Exercise 3(c) Simply divide each column of $\mathbf{P}$ by its length so that the new columns have length 1. Therefore $\mathbf{Q} = \Big[\frac{1}{\sqrt{3}}(\mathbf{P}_{|1});\ \frac{1}{\sqrt{2}}(\mathbf{P}_{|3});\ \frac{1}{\sqrt{6}}(\mathbf{P}_{|3});\Big] = \begin{bmatrix} \frac{\sqrt{3}}{3} & -\frac{\sqrt{2}}{2} & \frac{\sqrt{6}}{6} \\ \frac{\sqrt{3}}{3} & \frac{\sqrt{2}}{2} & \frac{\sqrt{6}}{6} \\ \frac{\sqrt{3}}{3} & 0 & -\frac{\sqrt{6}}{3} \end{bmatrix}$.

□

(Final June 22/23) Exercise 3(d) It is enough to show that $\mathbf{B}$ is symmetric.. We know that $\mathbf{A}$ is symmetric as it can be factored as $\mathbf{QDQ}^{\intercal}$. And because the inverse of a symmetric matrix is symmetric, we have that $\mathbf{B}^{\intercal} = (\mathbf{A} - \mathbf{A}^{-1})^{\intercal} = \mathbf{A}^{\intercal} - (\mathbf{A}^{-1})^{\intercal} = \mathbf{A} - \mathbf{A}^{-1} = \mathbf{B}$. So that,

$$q_{\mathbf{B}}(x,y,z) = \boldsymbol{x}\mathbf{B}\boldsymbol{x} = \boldsymbol{x}\mathbf{A}\boldsymbol{x} - \boldsymbol{x}(\mathbf{A}^{-1})\boldsymbol{x} = \boldsymbol{x}\mathbf{QDQ}^{\intercal}\boldsymbol{x} - \boldsymbol{x}\mathbf{Q}(\mathbf{D}^{-1})\mathbf{Q}^{\intercal}\boldsymbol{x} = \boldsymbol{x}\mathbf{Q}\big(\mathbf{D} - \mathbf{D}^{-1}\big)\mathbf{Q}^{\intercal}\boldsymbol{x};$$

that is to say

$$\boldsymbol{x}\mathbf{B}\boldsymbol{x} = \boldsymbol{x}\mathbf{Q}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{3}{2} \end{bmatrix}\mathbf{Q}^{\intercal}\boldsymbol{x} = \boldsymbol{x}\begin{bmatrix} \frac{\sqrt{3}}{3} & -\frac{\sqrt{2}}{2} & \frac{\sqrt{6}}{6} \\ \frac{\sqrt{3}}{3} & \frac{\sqrt{2}}{2} & \frac{\sqrt{6}}{6} \\ \frac{\sqrt{3}}{3} & 0 & -\frac{\sqrt{6}}{3} \end{bmatrix}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{3}{2} \end{bmatrix}\begin{bmatrix} \frac{\sqrt{3}}{3} & \frac{\sqrt{3}}{3} & \frac{\sqrt{3}}{3} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ \frac{\sqrt{6}}{6} & \frac{\sqrt{6}}{6} & -\frac{\sqrt{6}}{3} \end{bmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$= \begin{pmatrix} x, & y, & z, \end{pmatrix}\begin{bmatrix} 0 & 0 & \frac{\sqrt{6}}{4} \\ 0 & 0 & \frac{\sqrt{6}}{4} \\ 0 & 0 & -\frac{\sqrt{6}}{2} \end{bmatrix}\begin{pmatrix} \frac{\sqrt{3}(x+y+z)}{3} \\ \frac{\sqrt{2}(-x+y)}{2} \\ \frac{\sqrt{6}(x+y-2z)}{6} \end{pmatrix} = \begin{pmatrix} 0, & 0, & \frac{\sqrt{6}(x+y-2z)}{4}, \end{pmatrix}\begin{pmatrix} \frac{\sqrt{3}(x+y+z)}{3} \\ \frac{\sqrt{2}(-x+y)}{2} \\ \frac{\sqrt{6}(x+y-2z)}{6} \end{pmatrix}$$

$$= \frac{(x+y-2z)^2}{4}.$$

□

(Final June 22/23) Short questions set 1(a) Given that the matrix $\mathbf{B}^{\intercal}\mathbf{B}$ is diagonal and that ${}_{i|}(\mathbf{B}^{\intercal}\mathbf{B})_{|j} = (\mathbf{B}_{|i}) \cdot (\mathbf{B}_{|j})$, we known that $\mathbf{B}_{|i} \perp \mathbf{B}_{|j}$ when $i \neq j$ (the columns of $\mathbf{B}$ are orthogonal to each other). Furthermore, since the columns of $\mathbf{B}$ belong to $\mathbb{R}^3$, then $3 \geq \operatorname{rg}(\mathbf{B}) \geq \operatorname{rg}(\mathbf{B}^{\intercal}\mathbf{B}) = 3$. Therefore the 3 columns of $\mathbf{B}$ are a basis of $\mathbb{R}^3$ formed by vectors perpendicular to each other.

□

(Final June 22/23) Short questions set 1(b) $(\mathbf{A} - \beta\mathbf{I})\boldsymbol{x} = \mathbf{0}$ has a unique solution if and only if $(\mathbf{A} - \beta\mathbf{I})$ is invertible. Since the λ values that satisfy the characteristic equation are those that make the matrix $(\mathbf{A} - \lambda\mathbf{I})$ singular, the values of β have to be such that $\beta(\beta-1)(\beta^2-4) \neq 0$. In particular β has to be different from 0, different from 1, different from 2 and different from -2.

□

(Final June 22/23) Short questions set 1(c) Since the columns of $\mathbf{A}$ are linear combinations of the columns of $\mathbf{B}$, then $\mathcal{C}(\mathbf{A}) \subset \mathcal{C}(\mathbf{B})$. And $\mathbf{B}$ (with two columns) and $\mathbf{C}$ (with two rows) have rank 2, hence $\operatorname{rg}(\mathbf{A}) = \dim\mathcal{C}(\mathbf{A}) = \dim\mathcal{C}(\mathbf{B}) = 2$. Consequently, the columns of $\mathbf{B}$ form a basis of $\mathcal{C}(\mathbf{A})$.

□

(Final June 22/23) Short questions set 2(a)

$$\left[\begin{array}{ccc} 1 & 1 & 1 \\ 1 & 0 & 1 \\ \hline x & y & z \end{array}\right] \xrightarrow[\substack{[(-1)\mathbf{1}+\mathbf{2}] \\ [(-1)\mathbf{1}+\mathbf{3}]}]{\boldsymbol{\tau}} \left[\begin{array}{ccc} 1 & 0 & 0 \\ 1 & -1 & 0 \\ \hline x & -x+y & -x+z \end{array}\right] \Rightarrow \mathcal{H} = \left\{\boldsymbol{v} \in \mathbb{R}^3 \;\middle|\; \begin{bmatrix} -1 & 0 & 1 \end{bmatrix}\boldsymbol{v} = (0,)\right\}.$$

$$\left[\begin{array}{ccc} 3 & 2 & 4 \\ 0 & 0 & 1 \\ \hline x & y & z \end{array}\right] \xrightarrow{\substack{\tau \\ [(3)\mathbf{2}] \\ [(-2)\mathbf{1}+\mathbf{2}] \\ [(3)\mathbf{3}] \\ [(-4)\mathbf{1}+\mathbf{3}]}} \left[\begin{array}{ccc} 3 & 0 & 0 \\ 0 & 0 & 3 \\ \hline x & -2x+3y & -4x+3z \end{array}\right] \Rightarrow \mathcal{G} = \left\{ \boldsymbol{v} \in \mathbb{R}^3 \;\middle|\; \begin{bmatrix} -2 & 3 & 0 \end{bmatrix} \boldsymbol{v} = (0,) \right\}.$$

$\square$

(Final June 22/23) Short questions set 2(b) They are the vectors at the intersection of the planes $\mathcal{H}$ and $\mathcal{G}$. That is, they are those vectors that simultaneously satisfy their corresponding Cartesian equations. Thus, the requested vectors must satisfy the system of equations: $\begin{bmatrix} -2 & 3 & 0 \\ -1 & 0 & 1 \end{bmatrix} \boldsymbol{x} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$; therefore

$$\left[\begin{array}{ccc} -2 & 3 & 0 \\ -1 & 0 & 1 \\ \hline 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right] \xrightarrow{\substack{\tau \\ [(2)\mathbf{2}] \\ [(3)\mathbf{1}+\mathbf{2}]}} \left[\begin{array}{ccc} -2 & 0 & 0 \\ -1 & -3 & 1 \\ \hline 1 & 3 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{array}\right] \xrightarrow{\substack{\tau \\ [(3)\mathbf{3}] \\ [(1)\mathbf{2}+\mathbf{3}]}} \left[\begin{array}{ccc} -2 & 0 & 0 \\ -1 & -3 & 0 \\ \hline 1 & 3 & 3 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{array}\right] \Rightarrow \boxed{\mathcal{H} \cap \mathcal{G} = \mathcal{L}\left(\left[\begin{pmatrix} 3 \\ 2 \\ 3 \end{pmatrix}; \right] \right)}$$

$\square$

(Final June 22/23) Short questions set 3(a) $\begin{bmatrix} -1 & -1 & 0 & 0 \\ -1 & -2 & 0 & 0 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & -1 & -a \end{bmatrix} \xrightarrow{\overset{\tau}{[(-1)\mathbf{1}+\mathbf{2}]}} \begin{bmatrix} -1 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & -1 & -a \end{bmatrix} \xrightarrow{\overset{\tau}{[(-1)\mathbf{3}+\mathbf{4}]}}$ $\begin{bmatrix} -1 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & 1-a \end{bmatrix}$ therefore $\boxed{a \neq 1}$.

$\square$

(Final June 22/23) Short questions set 3(b) Since $\mathsf{P}(\mathsf{P}^{-1}) = \mathsf{I}$ we known that $\mathsf{P}\left(\mathsf{P}^{-1}\right)_{|j} = \mathsf{I}_{|j}$. Therefore, it is enough to multiply $\mathsf{P}\boldsymbol{v}$ and check if there are values of a for which $\mathsf{P}\boldsymbol{v}$ is the second column of the identity matrix.

$$\begin{bmatrix} -1 & -1 & 0 & 0 \\ -1 & -2 & 0 & 0 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & -1 & -a \end{bmatrix} \begin{pmatrix} 0 \\ 0 \\ -\frac{4}{3} \\ \frac{1}{3} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ \frac{4}{3} - \frac{a}{3} \end{pmatrix} \qquad \Rightarrow \qquad \text{when } a = 4, \text{ we get } \boldsymbol{v} = \left(\mathsf{P}^{-1}\right)_{|3}.$$

$\square$

(Final June 22/23) Short questions set 3(c) Necessarily $\boxed{a > 1}$. We can see it diagonalizing by congruence:

$$\begin{bmatrix} -1 & -1 & 0 & 0 \\ -1 & -2 & 0 & 0 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & -1 & -a \end{bmatrix} \xrightarrow[\overset{\tau}{[(-1)\mathbf{1}+\mathbf{2}]}]{\overset{\tau}{[(-1)\mathbf{1}+\mathbf{2}]}} \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & -1 & -a \end{bmatrix} \xrightarrow[\overset{\tau}{[(-1)\mathbf{3}+\mathbf{4}]}]{\overset{\tau}{[(-1)\mathbf{3}+\mathbf{4}]}} \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1-a \end{bmatrix}$$

or by leading principal minors, since if P is negative definite, then $-\mathsf{P} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & a \end{bmatrix}$ is positive definite: $\det\begin{bmatrix} 1 \end{bmatrix} = 1$; $\det\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} = 1$; $\det\begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} = 1$; $\det(-\mathsf{P}) = a - 1$.

$\square$

(Final June 22/23) Short questions set 4(a) It's false. For example if $\mathsf{A} = \mathbf{0}$, we always have that $\mathsf{B}\mathbf{0} = \mathbf{0}$.

$\square$

(Final June 22/23) Short questions set 4(b) It is impossible. Both A and D have rank 2 at most. The product cannot be rank 3.

$\square$

(Final May 22/23) Exercise 1(a) Since A has 5 columns and the dimension of its null space is 2: $\operatorname{rg}\left(\mathsf{A}\right) = 5 - 2 = 3$,

Given that the particular solution indicated in the parametric equations of the statement is the first column of the identity matrix of order 5, we have $\mathsf{A}\left(\mathsf{I}\right)_{|1} = \mathsf{A}_{|1} = \boldsymbol{b}$.

$\square$

(Final May 22/23) Exercise 1(b) Since $\mathbf{B}_{|j} = \mathbf{A}\mathbf{E}_{|j} = \mathbf{A}\big(\mathbf{E}_{|j}\big)$, each column of $\mathbf{B}$ is a linear combination of the columns of $\mathbf{A}$, that is, $\mathcal{C}\left(\mathbf{B}\right) \subset \mathcal{C}\left(\mathbf{A}\right)$. And since $\mathbf{A}_{|j} = \mathbf{B}(\mathbf{E}^{-1})_{|j} = \mathbf{B}\big(\mathbf{E}^{-1}{}_{|j}\big)$, each column of $\mathbf{A}$ is a linear combination of the columns of $\mathbf{B}$, ie $\mathcal{C}\left(\mathbf{A}\right) \subset \mathcal{C}\left(\mathbf{B}\right)$. Hence $\mathcal{C}\left(\mathbf{A}\right) = \mathcal{C}\left(\mathbf{B}\right)$.

□

(Final May 22/23) Exercise 1(c) Given that each row of $\mathbf{A}$ has 5 components and that the rank of $\mathbf{A}$ is 3, a basis of the row space of $\mathbf{A}$ consists of three linearly independent vectors of $\mathbb{R}^5$ which are perpendicular to the null space $\mathcal{N}\left(\mathbf{A}\right)$. The three given vectors belong to $\mathbb{R}^5$ and are linearly independent (just look at their last three components). They are also perpendicular to the two special solutions used in the parametric equations of the statement, therefore $\boxed{\text{those three vectors form a basis of } \mathcal{C}\left(\mathbf{A}^{\mathsf{T}}\right)}$.

□

(Final May 22/23) Exercise 1(d) It is enough to use the vectors of section (c) as rows of the coefficient matrix

$$\mathcal{N}\left(\mathbf{A}\right) = \left\{ \boldsymbol{x} \in \mathbb{R}^5 \;\middle|\; \begin{bmatrix} -1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \boldsymbol{x} = \mathbf{0} \right\}.$$

□

(Final May 22/23) Exercise 1(e) $\boxed{\boldsymbol{v} \in \mathcal{N}\left(\mathbf{A}\right)}$ because $\boldsymbol{v}$ satisfies the Cartesian equations found in the previous section: $\begin{bmatrix} -1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} -1 \\ -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

As a basis of $\mathcal{N}\left(\mathbf{A}\right)$ we can use the special solutions provided in the parametric equations of the problem statement. We only need to know which linear combination of the basis vectors is equal to $\boldsymbol{v}$:

$$\left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 1 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ \hline 1 & 0 & 0 \\ 0 & 1 & 0 \\ \hline 0 & 0 & 1 \end{array}\right] \xrightarrow[{[(-1)\mathbf{1}+\mathbf{3}]}]{\overset{\tau}{[(-1)\mathbf{1}+\mathbf{2}]}} \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 1 & -1 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ \hline 1 & -1 & -1 \\ 0 & 1 & 0 \\ \hline 0 & 0 & 1 \end{array}\right] \xrightarrow{\overset{\tau}{[(-2)\mathbf{2}+\mathbf{3}]}} \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ \hline 1 & -1 & 1 \\ 0 & 1 & -2 \\ \hline 0 & 0 & 1 \end{array}\right] \qquad \text{Coordinates: } \begin{pmatrix} 1 \\ -2 \end{pmatrix}.$$

□

(Final May 22/23) Exercise 2(a) Applying elimination by columns:

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & a & -b_1 \\ 1 & 1 & 0 & a & -b_2 \\ 0 & 1 & -1 & c & -b_3 \end{array}\right] \xrightarrow{\substack{\overset{\tau}{[(-1)\mathbf{1}+\mathbf{3}]} \\ [(-a)\mathbf{1}+\mathbf{4}] \\ [(b_1)\mathbf{1}+\mathbf{5}]}} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & -1 & 0 & b_1 - b_2 \\ 0 & 1 & -1 & c & -b_3 \end{array}\right] \xrightarrow{\substack{\overset{\tau}{[(1)\mathbf{2}+\mathbf{3}]} \\ [(-b_1+b_2)\mathbf{2}+\mathbf{5}]}} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & c & -b_1 + b_2 - b_3 \end{array}\right]$$

or, applying elimination by rows:

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & a & -b_1 \\ 1 & 1 & 0 & a & -b_2 \\ 0 & 1 & -1 & c & -b_3 \end{array}\right] \xrightarrow[\underset{\tau}{[(-1)\mathbf{1}+\mathbf{2}]}]{} \left[\begin{array}{cccc|c} 1 & 0 & 1 & a & -b_1 \\ 0 & 1 & -1 & 0 & b_1 - b_2 \\ 0 & 1 & -1 & c & -b_3 \end{array}\right] \xrightarrow[\underset{\tau}{[(-1)\mathbf{2}+\mathbf{3}]}]{} \left[\begin{array}{cccc|c} 1 & 0 & 1 & a & -b_1 \\ 0 & 1 & -1 & 0 & b_1 - b_2 \\ 0 & 0 & 0 & c & -b_1 + b_2 - b_3 \end{array}\right]$$

we conclude that whether $c \neq 0$ or $-b_1 + b_2 - b_3 = 0$ the system of equation is solvable.

□

(Final May 22/23) Exercise 2(b)

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & a & -2 \\ 1 & 1 & 0 & a & -1 \\ 0 & 1 & -1 & c & 1 \\ \hline 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{\substack{\overset{\tau}{[(-1)\mathbf{1}+\mathbf{3}]} \\ [(-a)\mathbf{1}+\mathbf{4}] \\ [(2)\mathbf{1}+\mathbf{5}]}} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & -1 & 0 & 1 \\ 0 & 1 & -1 & c & 1 \\ \hline 1 & 0 & -1 & -a & 2 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{\substack{\overset{\tau}{[(1)\mathbf{2}+\mathbf{3}]} \\ [(-1)\mathbf{2}+\mathbf{5}]}} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & c & 0 \\ \hline 1 & 0 & -1 & -a & 2 \\ 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 \end{array}\right].$$

When $c \neq 0$ the solution set is $\left\{ \boldsymbol{v} \in \mathbb{R}^4 \;\middle|\; \exists \boldsymbol{p} \in \mathbb{R}^1,\ \boldsymbol{v} = \begin{pmatrix} 2 \\ -1 \\ 0 \\ 0 \end{pmatrix} + \begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \end{bmatrix} \boldsymbol{p} \right\}$.

And when $c = 0$ the solution set is $\left\{ \boldsymbol{v} \in \mathbb{R}^4 \;\middle|\; \exists \boldsymbol{p} \in \mathbb{R}^2,\ \boldsymbol{v} = \begin{pmatrix} 2 \\ -1 \\ 0 \\ 0 \end{pmatrix} + \begin{bmatrix} -1 & -a \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \boldsymbol{p} \right\}$.

□

(Final May 22/23) Exercise 2(c) The subspace of solutions of $\mathbf{A}\boldsymbol{x} = \mathbf{0}$ must be of dimension 2. So $c = 0$. Furthermore, the vectors of B have to be perpendicular to the rows of $\mathbf{A}$. Therefore, the product of $\mathbf{A}$ by the matrix $\mathbf{B}$ whose columns are the vectors of B *must be a zero matrix*:

$$\begin{bmatrix} 1 & 0 & 1 & a \\ 1 & 1 & 0 & a \\ 0 & 1 & -1 & c \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 1 & 1 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & a+1 \\ 0 & a+1 \\ 0 & c \end{bmatrix} \quad \Rightarrow \quad \begin{cases} c = 0 \\ a = -1 \end{cases}.$$

□

(Final May 22/23) Exercise 2(d) Basta ver cómo deben ser los parámetros para que las filas de $\mathbf{A}$ sean combinación lineal de las filas de $\mathbf{C}$. Podemos hacerlo mediante eliminación por columnas (transponiendo las matrices $\mathbf{C}$ y $\mathbf{A}$)It is enough to see how the parameters must be so that the rows of $\mathbf{A}$ are a linear combination of the rows of $\mathbf{C}$. We can do this by column elimination (transposing the matrices $\mathbf{C}$ and $\mathbf{A}$).

$$\left[\begin{array}{cc|ccc} 1 & 1 & 1 & 1 & 0 \\ -1 & -2 & 0 & 1 & 1 \\ 2 & 3 & 1 & 0 & -1 \\ 4 & 4 & a & a & c \end{array}\right] \xrightarrow{\substack{\tau \\ [(-1)\mathbf{1}+\mathbf{2}] \\ [(-1)\mathbf{1}+\mathbf{3}] \\ [(-1)\mathbf{1}+\mathbf{4}]}} \left[\begin{array}{cc|ccc} 1 & 0 & 0 & 0 & 0 \\ -1 & -1 & 1 & 2 & 1 \\ 2 & 1 & -1 & -2 & -1 \\ 4 & 0 & a-4 & a-4 & c \end{array}\right] \xrightarrow{\substack{\tau \\ [(1)\mathbf{2}+\mathbf{3}] \\ [(2)\mathbf{2}+\mathbf{4}] \\ [(1)\mathbf{2}+\mathbf{5}]}} \left[\begin{array}{cc|ccc} 1 & 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 \\ 4 & 0 & a-4 & a-4 & c \end{array}\right],$$

or by row elimination:

$$\left[\begin{array}{cccc} 1 & -1 & 2 & 4 \\ 1 & -2 & 3 & 4 \\ \hline 1 & 0 & 1 & a \\ 1 & 1 & 0 & a \\ 0 & 1 & -1 & c \end{array}\right] \xrightarrow[\substack{\tau \\ [(-1)\mathbf{1}+\mathbf{4}] \\ [(-1)\mathbf{1}+\mathbf{3}] \\ [(-1)\mathbf{1}+\mathbf{2}]}]{} \left[\begin{array}{cccc} 1 & -1 & 2 & 4 \\ 0 & -1 & 1 & 0 \\ \hline 0 & 1 & -1 & a-4 \\ 0 & 2 & -2 & a-4 \\ 0 & 1 & -1 & c \end{array}\right] \xrightarrow[\substack{\tau \\ [(1)\mathbf{2}+\mathbf{5}] \\ [(2)\mathbf{2}+\mathbf{4}] \\ [(1)\mathbf{2}+\mathbf{3}]}]{} \left[\begin{array}{cccc} 1 & -1 & 2 & 4 \\ 0 & -1 & 1 & 0 \\ \hline 0 & 0 & 0 & a-4 \\ 0 & 0 & 0 & a-4 \\ 0 & 0 & 0 & c \end{array}\right]. \text{ so}$$

The answer is $\boxed{a = 4 \text{ and } c = 0}$.

□

(Final May 22/23) Exercise 3(a) Since $\mathbf{A}_{\tau_1 \cdots \tau_k} = \mathbf{A}\left(\mathbf{I}_{\tau_1 \cdots \tau_k}\right) = \mathbf{I} \quad \Rightarrow \quad \mathbf{I}_{\tau_1 \cdots \tau_k} = \mathbf{A}^{-1}$. It is enough to check that by elementary transformations we get the identity matrix:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 2 & b & 0 & 1 \end{bmatrix} \xrightarrow{\substack{\tau \\ [(-1)\mathbf{2}+\mathbf{1}]}} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 2-b & b & 0 & 1 \end{bmatrix} \xrightarrow{\substack{\tau \\ [(b-2)\mathbf{4}+\mathbf{1}] \\ [(-b)\mathbf{4}+\mathbf{2}]}} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{\substack{\tau \\ [(\frac{1}{2})\mathbf{2}]}} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

□

(Final May 22/23) Exercise 3(b) Given that if the columns of the full rank matrix $\mathbf{S}$ are eigenvectors of $\mathbf{A}$ we have that $\mathbf{A} = \mathbf{SD}(\mathbf{S}^{-1})$, then if $\mathbf{A}$ is full rank, we have that $\mathbf{A}^{-1} = \left(\mathbf{SD}(\mathbf{S}^{-1})\right)^{-1} = \mathbf{SD}^{-1}(\mathbf{S}^{-1})$; that is, $\mathbf{A}^{-1}$ is also diagonalizable, it has the same eigenvectors as $\mathbf{A}$ and its eigenvalues are the inverse of those of $\mathbf{A}$. So we can deal directly with $\mathbf{A}$. As the eigenvectors of $\mathbf{A}$ are the numbers on the diagonal (because $\mathbf{A}$ is triangular), we can see that $\lambda = 1$ has algebraic multiplicity 3, let's see in which cases $\mathbf{A}$ (and thus $\mathbf{A}^{-1}$) is diagonalizable:

$$\left[\begin{array}{c} \mathbf{A} - (1)\mathbf{I} \\ \hline \mathbf{I} \end{array}\right] = \left[\begin{array}{cccc} 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 2 & b & 0 & 0 \\ \hline 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{\substack{\tau \\ [(2)\mathbf{2}] \\ [(-1)\mathbf{1}+\mathbf{2}]}} \left[\begin{array}{cccc} 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 2 & 2(b-1) & 0 & 0 \\ \hline 1 & -1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right],$$

so, for the dimension of the eigenspace $\mathcal{E}_{\lambda=1}$ to be 3, the parameter must be $\boxed{b=1}$.

$\square$

(Final May 22/23) Exercise 3(c) We already know that $\left[\begin{pmatrix}-1\\2\\0\\0\end{pmatrix};\ \begin{pmatrix}0\\0\\1\\0\end{pmatrix};\ \begin{pmatrix}0\\0\\0\\1\end{pmatrix};\right]$ is a basis of $\mathcal{E}_{\lambda=1}$ when $b=1$.

On the other hand, since $\left[\frac{\mathbf{A}-(2)\mathbf{I}}{\mathbf{I}}\right] = \left[\begin{array}{cccc}-1&0&0&0\\2&0&0&0\\0&0&-1&0\\2&1&0&-1\\\hline 1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{array}\right] \xrightarrow{\underset{[(1)\mathbf{2}+\mathbf{4}]}{\boldsymbol{\tau}}} \left[\begin{array}{cccc}-1&0&0&0\\2&0&0&0\\0&0&-1&0\\2&1&0&0\\\hline 1&0&0&0\\0&1&0&1\\0&0&1&0\\0&0&0&1\end{array}\right]$, then

$\left[\begin{pmatrix}0\\1\\0\\1\end{pmatrix};\ \begin{pmatrix}-1\\2\\0\\0\end{pmatrix};\ \begin{pmatrix}0\\0\\1\\0\end{pmatrix};\ \begin{pmatrix}0\\0\\0\\1\end{pmatrix};\right]$ is a basis of $\mathbb{R}^4$ consisting of eigenvectors of $\mathbf{A}$.

$\square$

(Final May 22/23) Exercise 3(d)

- $|2\mathbf{A}^{\intercal}\mathbf{B}| = |2\mathbf{A}||\mathbf{B}| = 2^4\times 2\times 2 = 2^6 = 64$.
- $|\mathbf{AB}-\mathbf{B}| = |(\mathbf{A}-\mathbf{I})\mathbf{B}| = 0\times 2 = 0$.
- $\operatorname{tr}\left(\mathbf{BAB}^{-1}\right) = \operatorname{tr}\left(\mathbf{A}\right) = 5$ since $\mathbf{BAB}^{-1}$ is similar to $\mathbf{A}$.

$\square$

(Final May 22/23) Short questions set 1(a) **True**: Since $\mathbf{AB}_{|j} = \mathbf{BD}_{|j} = \mathbf{B}\lambda_j(\mathbf{I}_{|j}) = \lambda_j\mathbf{B}_{|j}$, where $\mathbf{D}$ is a diagonal matrix, whose diagonal contains the corresponding eigenvalues λ_j, and since $\boldsymbol{x} = \mathbf{B}\boldsymbol{a}$, then

$$\mathbf{A}\boldsymbol{x} = \mathbf{AB}\boldsymbol{a} = \mathbf{BD}\boldsymbol{a} = \mathbf{B}\begin{pmatrix}\lambda_1 a_1\\ \vdots\\ \lambda_n a_n\end{pmatrix}.$$

Another proof: Since the columns of $\mathbf{B}$ are a basis of $\mathbb{R}^n$, then $\mathbf{A} = \mathbf{BDB}^{-1}$, where $\mathbf{D}$ is diagonal with the correspondent aigenvalues in its main diagonal, so $\mathbf{D}\boldsymbol{a} = (\lambda_1 a_1, \ldots \lambda_n a_n,)$. Therefore, since $\boldsymbol{x} = \mathbf{B}\boldsymbol{a}$, then $\mathbf{A}\boldsymbol{x} = \mathbf{BDB}^{-1}\boldsymbol{x} = \mathbf{BDB}^{-1}\mathbf{B}\boldsymbol{a} = \mathbf{B}(\mathbf{D}\boldsymbol{a})$.

$\square$

(Final May 22/23) Short questions set 1(b) **False**: Let $B = \left\{\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}\right\}$ (that is, a basis for the subspace of vectors whose **last** component is zero) and let $\mathbf{A} = \begin{bmatrix}0&0&1\\0&1&0\\1&0&0\end{bmatrix}$; then,

$$B^* = \left\{\begin{bmatrix}0&0&1\\0&1&0\\1&0&0\end{bmatrix}\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{bmatrix}0&0&1\\0&1&0\\1&0&0\end{bmatrix}\begin{pmatrix}0\\1\\0\end{pmatrix}\right\} = \left\{\begin{pmatrix}0\\0\\1\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}\right\}$$

is a basis for the subspace of vectors whose **first** component is zero.

$\square$

(Final May 22/23) Short questions set 1(c) **True**: Since ${}_{j|}\mathbf{C} = {}_{j|}(\mathbf{A}+\mathbf{A}^{\intercal}) = {}_{j|}\mathbf{A} + {}_{j|}(\mathbf{A}^{\intercal}) = (\mathbf{A}^{\intercal})_{|j} + \mathbf{A}_{|j} = \mathbf{C}_{|j}$, The matrix $\mathbf{C}$ is symmetric and, therefore, diagonalizable.

$\square$

(Final May 22/23) Short questions set 1(d) **True**: As $\mathbf{A}\boldsymbol{v} = 3\boldsymbol{v}$, we have that $12 = f(\boldsymbol{v}) = \boldsymbol{v}\mathbf{A}\boldsymbol{v} = \boldsymbol{v}\cdot(3\boldsymbol{v}) = 3(\boldsymbol{v}\cdot\boldsymbol{v}) = 3\|\boldsymbol{v}\|^2$; so $\|\boldsymbol{v}\|^2 = 4$; hence $\boxed{\|\boldsymbol{v}\| = 2}$.

$\square$

(Final May 22/23) Short questions set 1(e) **True**: Since $1 = |\mathbf{A}| = \lambda_1 \times \lambda_2 \times \lambda_3 = 1 \times 2 \times \lambda_3$, we have $\lambda_3 = \frac{1}{2}$. And since the eigenvalues of $\mathbf{A}^{\text{-}1}$ are the inverses of the eigenvalues of $\mathbf{A}$, we know that $\mathbf{A}$ and $\mathbf{A}^{\text{-}1}$ have the same eigenvalues. Therefore, they have the same trace.

$\square$

(Final May 22/23) Short questions set 2(a) Since $\operatorname{rg}\big(\underset{3\times 3}{\mathbf{A}(\mathbf{B}^{\intercal})}\big) = 3$, it is necessary that $\operatorname{rg}\big(\underset{3\times 4}{\mathbf{A}}\big) = \operatorname{rg}\big(\underset{3\times 4}{\mathbf{B}}\big) = 3$.

$\square$

(Final May 22/23) Short questions set 2(b) On the one hand, $\Big|\underset{4\times 4}{(\mathbf{B}^{\intercal})\mathbf{B}}\Big| = 0 = \Big|\underset{4\times 4}{(\mathbf{B}^{\intercal})\mathbf{A}}\Big|$ (since the rank can't be 4), on the other hand $|\mathbf{B}(\mathbf{A}^{\intercal})| = \big|(\mathbf{B}(\mathbf{A}^{\intercal}))^{\intercal}\big| = |\mathbf{A}(\mathbf{B}^{\intercal})| = 2$.

$\square$

(Final May 22/23) Short questions set 2(c) Given that $\mathbf{A}(\mathbf{B}^{\intercal})$ has full rank (invertible), the only solution is $\boldsymbol{x} = \mathbf{0}$.

$\square$

(Final May 22/23) Short questions set 2(d)

$$\begin{aligned}\det\big[\mathbf{C}_{|2};\ (\mathbf{C}_{|2} - \mathbf{C}_{|1});\ (2\mathbf{C}_{|2} + \mathbf{C}_{|3});\ \big] &= \det\big[\mathbf{C}_{|2};\ -\mathbf{C}_{|1};\ \mathbf{C}_{|3};\ \big] = -\det\big[\mathbf{C}_{|2};\ \mathbf{C}_{|1};\ \mathbf{C}_{|3};\ \big]\\ &= \det\big[\mathbf{C}_{|1};\ \mathbf{C}_{|2};\ \mathbf{C}_{|3};\ \big] = |\mathbf{C}| = |\mathbf{A}\mathbf{B}^{\intercal}| = 2.\end{aligned}$$

$\square$

(Final May 22/23) Short questions set 3(a) Diagonalizing by congruence, we can determine the signs of the eigenvalues. Hence, since $\begin{bmatrix} 1 & -1 & 0 \\ -1 & b & 0 \\ 0 & 0 & 2 \end{bmatrix} \xrightarrow{\overset{\boldsymbol{\tau}}{[(1)\mathbf{1}+\mathbf{2}]}} \begin{bmatrix} 1 & 0 & 0 \\ -1 & b-1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \xrightarrow[{\overset{\boldsymbol{\tau}}{[(1)\mathbf{1}+\mathbf{2}]}}]{} \begin{bmatrix} 1 & 0 & 0 \\ 0 & b-1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ we conclude that

- when $b < 1$, it can take both positive and negative values.
- when $b = 1$, it is positive semidefinite.
- when $b > 1$, it is negative semidefinite.

$\square$

(Final July 21/22) Exercise 1(a) $\begin{bmatrix} 1 & 0 & 2 & 3 \\ -2 & 2 & 0 & 2 \\ 0 & 1 & 0 & 0 \\ 1 & 3 & 1 & 0 \end{bmatrix} \xrightarrow{\substack{\boldsymbol{\tau} \\ [(-2)\mathbf{1}+\mathbf{3}] \\ [(-3)\mathbf{1}+\mathbf{4}]}} \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 2 & 4 & 8 \\ 0 & 1 & 0 & 0 \\ 1 & 3 & -1 & -3 \end{bmatrix} \xrightarrow{\substack{\boldsymbol{\tau} \\ [(-2)\mathbf{2}+\mathbf{3}] \\ [(-4)\mathbf{2}+\mathbf{4}]}} \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 2 & 0 & 0 \\ 0 & 1 & -2 & -4 \\ 1 & 3 & -7 & -15 \end{bmatrix} \xrightarrow{\text{[(}}$ $\begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 2 & 0 & 0 \\ 0 & 1 & -2 & 0 \\ 1 & 3 & -7 & -1 \end{bmatrix}$ therefore, they are 4 linearly independent vectors of $\mathbb{R}^4$.

$\square$

(Final July 21/22) Exercise 1(b) From above we have $\mathbf{A}\begin{bmatrix} 1 & 0 & 2 & 3 \\ -2 & 2 & 0 & 2 \\ 0 & 1 & 0 & 0 \\ 1 & 3 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 5 & 1 \\ 4 & 0 & 10 & 2 \end{bmatrix}$. Hence, if $\mathbf{B} = \begin{bmatrix} 1 & 0 & 2 & 3 \\ -2 & 2 & 0 & 2 \\ 0 & 1 & 0 & 0 \\ 1 & 3 & 1 & 0 \end{bmatrix}$ and $\mathbf{C} = \begin{bmatrix} 2 & 0 & 5 & 1 \\ 4 & 0 & 10 & 2 \end{bmatrix}$, then $\mathbf{A} = \mathbf{C}(\mathbf{B}^{\text{-}1})$, since, from the first section, we know that $\mathbf{B}$ is invertible.

$\square$

(Final July 21/22) Exercise 1(c) Since $\mathbf{A}^{\intercal} = \big(\mathbf{C}(\mathbf{B}^{\text{-}1})\big)^{\intercal} = (\mathbf{B}^{\text{-}1})^{\intercal}(\mathbf{C}^{\intercal})$, where $(\mathbf{B}^{\text{-}1})^{\intercal} = \big(\mathbf{B}^{\intercal}\big)^{\text{-}1}$ is full rank; and since $(\mathbf{A}^{\intercal})\,\boldsymbol{x} = \mathbf{0}$ if and only if $\mathbf{E}(\mathbf{A}^{\intercal})\boldsymbol{x} = \mathbf{E}\mathbf{0} = \mathbf{0}$, then, the solution set of $(\mathbf{A}^{\intercal})\,\boldsymbol{x} = \mathbf{0}$ is the same as the solution set of $\mathbf{E}(\mathbf{A}^{\intercal})\boldsymbol{x} = \mathbf{0}$. Thus, taking $\mathbf{E} = \mathbf{B}^{\intercal}$ we have that the set of vectors that are solution of $(\mathbf{A}^\intercal)\,\boldsymbol{x} = \mathbf{0}$ is the same as the set of vectors that are solution of $\mathbf{B}^\intercal(\mathbf{A}^\intercal)\boldsymbol{x} = \mathbf{B}^\intercal\Big((\mathbf{B}^\intercal)^{-1}(\mathbf{C}^\intercal)\Big)\boldsymbol{x} = (\mathbf{C}^\intercal)\,\boldsymbol{x} = \mathbf{0}$. So

$$\left[\begin{array}{cc} 2 & 4 \\ 0 & 0 \\ 5 & 10 \\ 1 & 2 \\ \hline 1 & 0 \\ 0 & 1 \end{array}\right] \xrightarrow{\overset{\tau}{[(-2)\mathbf{1}+\mathbf{2}]}} \left[\begin{array}{cc} 2 & 0 \\ 0 & 0 \\ 5 & 0 \\ 1 & 0 \\ \hline 1 & -2 \\ 0 & 1 \end{array}\right] \quad \Rightarrow \quad \text{a basis of } \mathcal{N}\left(\mathbf{A}^\intercal\right) \text{ is: } \left[\begin{pmatrix} -2 \\ 1 \end{pmatrix};\right]$$

$\square$

(Final July 21/22) Exercise 1(d) Since $\mathbf{A} = \underset{2\times 4}{\mathbf{C}}\,\underset{4\times 4}{(\mathbf{B}^{-1})}$ we must have $m = 2$ and $n = 4$. From above, the dimension of $\mathcal{N}\left(\mathbf{A}^\intercal\right)$ is 1, but this must equal $m - r$, so we obtain $r = 1$.

$\square$

(Final July 21/22) Exercise 2(a) $\mathbf{A}^\intercal\mathbf{A} = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$.

Since off-diagonal components are null $({}_{i|}\mathbf{A}^\intercal\mathbf{A}_{|j} = 0$ with $i \neq j)$, columns of $\mathbf{A}$ are orthogonal to each other.

$\square$

(Final July 21/22) Exercise 2(b) $\left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ -1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{\overset{\tau}{[(-1)\mathbf{1}+\mathbf{3}]}} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ -1 & 0 & 2 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{\overset{\tau}{[(-1)\mathbf{2}+\mathbf{4}]}}$

$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 2 & 0 & 0 \\ 0 & -1 & 0 & 2 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 \end{array}\right]$. So, $\boxed{\det\mathbf{A} = 4}$

$\square$

(Final July 21/22) Exercise 2(c) Since $\mathbf{A}^\intercal\mathbf{A} = 2\mathbf{I}$ then $\left(\frac{1}{2}\mathbf{A}^\intercal\right)\mathbf{A} = \mathbf{I}$.

$\square$

(Final July 21/22) Exercise 2(d) Since $\mathcal{S}$ is a subspace, the vectors in $\mathcal{S}$ must be solutions of a homogeneous system of equations. The rows of the coefficient matrix of that homogeneous system must be orthogonal to the first two columns of $\mathbf{A}$. Since the last two columns of $\mathbf{A}$ are perpendicular to the first two columns, we have that

$$\mathcal{S} = \left\{\boldsymbol{v} \in \mathbb{R}^4 \;\middle|\; \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}\boldsymbol{v} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}\right\}.$$

$\square$

(Final July 21/22) Exercise 2(e)

$$\mathbf{A}^9 = (\mathbf{A}^4)(\mathbf{A}^4)\mathbf{A} = (-4\mathbf{I})(-4\mathbf{I})\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 16 & 0 & 0 & 0 \\ 0 & 16 & 0 & 0 \\ 0 & 0 & 16 & 0 \\ 0 & 0 & 0 & 16 \end{bmatrix}\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 16 & 0 & 16 & 0 \\ 0 & 16 & 0 & 16 \\ -16 & 0 & 16 & 0 \\ 0 & -16 & 0 & 16 \end{bmatrix}.$$

$\square$

(Final July 21/22) Exercise 3(a) $\mathbf{A}^\intercal\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{bmatrix} a & 1-a \\ 1-b & b \end{bmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$; therefore the associated eigenvalue is $\lambda = 1$. On the other hand $(1,\quad 1,)$ is not an eigenvector of $\mathbf{A}$ unless $\mathbf{A}$ is symmetric because

$$\text{If}\quad \mathbf{A}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} a-b+1 \\ -a+b+1 \end{pmatrix} \text{ is a multiple of } \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad\Rightarrow\quad a-b = -a+b \quad\Rightarrow\quad 2a = 2b.$$

$\square$

(Final July 21/22) Exercise 3(b) On the one hand, since the components of $\mathbf{A}$ are non-negative, and since the components of $\boldsymbol{v}$ are also non-negative: $v_i \geq 0$

$${}_{i|}(\mathbf{A}\boldsymbol{v}) = {}_{i|}\Big((\mathbf{A}_{|1})v_1 + (\mathbf{A}_{|2})v_2\Big) = ({}_{i|}\mathbf{A}_{|1})v_2 + ({}_{i|}\mathbf{A}_{|2})v_2 \geq 0$$

since it is a sum of non-negative numbers. On the other hand, since $v_1+v_2=1$ and since the dot product of a vector $\boldsymbol{v}$ by a vector of ones is the sum of the components of $\boldsymbol{v}$, we have that

$$\boldsymbol{v}\cdot\begin{pmatrix}1\\1\end{pmatrix}=\begin{pmatrix}v_1\\v_2\end{pmatrix}\cdot\begin{pmatrix}1\\1\end{pmatrix}=v_1+v_2=1.$$

Thus, since $\mathbf{A}^\top\begin{pmatrix}1\\1\end{pmatrix}=\begin{pmatrix}1\\1\end{pmatrix}$; and since $\mathbf{A}\boldsymbol{v}=\boldsymbol{v}(\mathbf{A}^\top)$, the sum of the components of $\mathbf{A}\boldsymbol{v}$ is: $\boldsymbol{v}(\mathbf{A}^\top)\begin{pmatrix}1\\1\end{pmatrix}=\boldsymbol{v}\cdot\begin{pmatrix}1\\1\end{pmatrix}=1.$

□

(Final July 21/22) Exercise 3(c) The trace of $\mathbf{A}$ is $(a+b)$ and hence the sum of eigenvalues is $1+\lambda_2=a+b$; that is, $\boxed{\lambda_2=a+b-1}$, where $0\le a\le 1$ and $0\le b\le 1$. Thus, the extreme cases are:

$$\begin{cases}\lambda_2=1 & \text{when } a=b=1,\text{ in this case } \mathbf{A}=\begin{bmatrix}1&0\\0&1\end{bmatrix}\\[2ex] \lambda_2=-1 & \text{when } a=b=0,\text{ in this case } \mathbf{A}=\begin{bmatrix}0&1\\1&0\end{bmatrix}\end{cases},$$

in all other cases $-1<\lambda_2<1$. Therefore, in all cases $|\lambda_2|\le 1$.

□

(Final July 21/22) Exercise 3(d) Since $\mathbf{A}$ is symmetric we alredy known (part a) that $\begin{pmatrix}1, & 1,\end{pmatrix}$ is an eigenvector, so any nonzero perpendicular vector in $\mathbb{R}^2$ is another eigenvector, for example $\begin{pmatrix}-1, & 1,\end{pmatrix}$. But we can compute them:

$$\begin{cases}\text{For } \lambda_1=1: & \mathbf{A}-\mathbf{I}=\begin{bmatrix}a-1&1-a\\1-a&a-1\end{bmatrix}\implies\mathcal{E}_{(1)}(\mathbf{A})=\mathcal{L}\left(\left[\begin{pmatrix}1\\1\end{pmatrix};\right]\right)\\[2ex] \text{For } \lambda_2=2a-1: & \mathbf{A}-(2a-1)\mathbf{I}=\begin{bmatrix}1-a&1-a\\1-a&1-a\end{bmatrix}\implies\mathcal{E}_{(2a-1)}(\mathbf{A})=\mathcal{L}\left(\left[\begin{pmatrix}-1\\1\end{pmatrix};\right]\right).\end{cases}$$

Therefore, $\mathsf{B}=\left[\begin{pmatrix}1\\1\end{pmatrix};\begin{pmatrix}-1\\1\end{pmatrix};\right]$ is a basis of $\mathbb{R}^2$ that consists of eigenvectors of $\mathbf{A}$.

□

(Final July 21/22) Exercise 3(e) $\mathbf{A}^k\boldsymbol{x}=\mathbf{A}^k(\alpha\boldsymbol{v}+\beta\boldsymbol{w})=\alpha(\mathbf{A}^k)\boldsymbol{v}+\beta(\mathbf{A}^k)\boldsymbol{w}=\alpha(\lambda_1^k)\boldsymbol{v}+\beta(\lambda_2^k)\boldsymbol{w}$. Since $\lambda_1=1$ and since $|\lambda_2|<1$, then: $\lim_{k\to\infty}\mathbf{A}^k\boldsymbol{x}=\alpha\boldsymbol{v}=\alpha\begin{pmatrix}1\\1\end{pmatrix}$. And since the elements of $\boldsymbol{z}$ add up to 1, then $\alpha=\frac{1}{2}$, therefore

$$\boldsymbol{z}=\lim_{k\to\infty}\mathbf{A}^k\boldsymbol{x}=\frac{1}{2}\begin{pmatrix}1\\1\end{pmatrix}.$$

□

(Final July 21/22) Short questions set 1(a) Such a matrix must be an m matrix whose nullspace is one dimensional. In other words, the rank is $3-1=2$. We may take $\mathbf{A}$ to be an 2×3 matrix whose rows are linearly independent. As an example, we take

$$\mathbf{A}=\begin{bmatrix}1&1&1\\1&1&2\end{bmatrix},\qquad \boldsymbol{b}=\begin{pmatrix}1\\1\end{pmatrix}.$$

□

(Final July 21/22) Short questions set 1(b) To find all solutions, we do elimination:

$$\left[\begin{array}{ccc|c}1&1&1&-1\\1&1&2&-1\\\hline 1&0&0&0\\0&1&0&0\\0&0&1&0\\\hline 0&0&0&1\end{array}\right]\xrightarrow[{[(1)\mathbf{1}+\mathbf{4}]}]{\substack{\boldsymbol{\tau}\\ [(-1)\mathbf{1}+\mathbf{2}]\\ [(-1)\mathbf{1}+\mathbf{3}]}}\left[\begin{array}{ccc|c}1&0&0&0\\1&0&1&0\\\hline 1&-1&-1&1\\0&1&0&0\\0&0&1&0\\\hline 0&0&0&1\end{array}\right]\Rightarrow\left\{\boldsymbol{v}\in\mathbb{R}^3\;\middle|\;\exists\boldsymbol{p}\in\mathbb{R}^1,\ \boldsymbol{v}=\begin{pmatrix}1\\0\\0\end{pmatrix}+\begin{bmatrix}-1\\1\\0\end{bmatrix}\boldsymbol{p}\right\}.$$

□

(Final July 21/22) Short questions set 2(a) We will use the upper left determinants:

$$\det\begin{bmatrix} a \end{bmatrix} = a > 0;\quad \det\begin{bmatrix} a & 2 \\ 2 & a \end{bmatrix} = (a-2)(a+2) > 0;\quad \det\begin{bmatrix} a & 2 & 1 \\ 2 & a & 1 \\ 1 & 1 & 2 \end{bmatrix} = 2(a-2)(a+1) > 0.$$

So the condition is $a > 2$.

□

(Final July 21/22) Short questions set 2(b) Since there is a positive element in the main diagonal, A could never be negative definite.

□

(Final July 21/22) Short questions set 2(c) Since $|\mathsf{A}| = 2(a-2)(a+1) \quad\longrightarrow\quad \mathsf{A}$ is singular if $a = -1$ or $a = 2$.

□

(Final July 21/22) Short questions set 3(a) A is upper triangular, so the eigenvalues are the entries in the diagonal: $0, 0, 0, 0$.

□

(Final July 21/22) Short questions set 3(b) A has rank 3, so there is only one linearly independent eigenvector. $\left[\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix};\right]$.

□

(Final July 21/22) Short questions set 4(a) $(\mathsf{A}^{\intercal})^{-1} = (\mathsf{A}^{-1})^{\intercal} = \begin{bmatrix} 4 & 3 & 3 \\ -1 & -1 & -1 \\ -3 & 0 & 1 \end{bmatrix}^{\intercal} = \boxed{\begin{bmatrix} 4 & -1 & -3 \\ 3 & -1 & 0 \\ 3 & -1 & 1 \end{bmatrix}}$.

□

(Final July 21/22) Short questions set 5(a) **False.** If the solution set of $\mathsf{A}\boldsymbol{x} = \mathbf{0}$ is a line, then the rank of A is 2. Therefore, the only solution of $(\mathsf{A}^{\intercal})\boldsymbol{y} = \mathbf{0}$ is the point $\boldsymbol{y} = \mathbf{0}$.

□

(Final July 21/22) Short questions set 5(b) **True.** $\mathsf{A} = \mathsf{PD}(\mathsf{P})^{-1} \quad\Rightarrow\quad \mathsf{A}^{\intercal} = (\mathsf{P}^{-1})^{\intercal}\mathsf{DP}^{\intercal} = (\mathsf{P}^{\intercal})^{-1}\mathsf{DP}^{\intercal}$.

□

(Final May 21/22) Exercise 1(a)

$$\left[\begin{array}{cccc} 1 & 2 & 1 & 0 \\ 2 & 5 & 1 & 1 \\ 0 & 1 & 1 & 3 \\ \hline 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow[{[(-1)\mathbf{1}+\mathbf{3}]}]{\overset{\tau}{[(-2)\mathbf{1}+\mathbf{2}]}} \left[\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 2 & 1 & -1 & 1 \\ 0 & 1 & 1 & 3 \\ \hline 1 & -2 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow[{[(-1)\mathbf{2}+\mathbf{4}]}]{\overset{\tau}{[(1)\mathbf{2}+\mathbf{3}]}} \left[\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 0 & 1 & 2 & 2 \\ \hline 1 & -2 & -3 & 2 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{\overset{\tau}{[(-1)\mathbf{3}+\mathbf{4}]}} \left[\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ \hline 1 & -2 & -3 & 5 \\ 0 & 1 & 1 & -2 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{array}\right].$$

Since the rank of A is 3, then $\mathbb{R}^3 \subset \mathcal{C}(\mathsf{A})$ and therefore $\mathsf{A}\boldsymbol{x} = \boldsymbol{b}$ will have infinitely many solutions for any $\boldsymbol{b} \in \mathbb{R}^3$ (since there are free columns). But since $\mathbb{R}^4 \not\subset \mathcal{C}(\mathsf{A}^{\intercal})$, then $\mathsf{A}^{\intercal}\boldsymbol{y} = \boldsymbol{c}$ may have no solution for some $\boldsymbol{c} \in \mathbb{R}^4$, but if it does, it will be unique (since there are no free rows in A).

□

(Final May 21/22) Exercise 1(b)

$$\begin{bmatrix} 1 & 2 & 1 & 0 \\ 2 & 5 & 1 & 1 \\ 0 & 1 & 1 & 3 \end{bmatrix} \xrightarrow[{\overset{\tau}{[(-2)\mathbf{1}+\mathbf{2}]}}]{} \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 1 & 1 & 3 \end{bmatrix} \xrightarrow[{\overset{\tau}{[(-1)\mathbf{2}+\mathbf{3}]}}]{} \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 2 & 2 \end{bmatrix}.$$

Since the last row belongs to $\mathcal{C}(\mathsf{A}^{\intercal})$, and only two of its components are non-zero, this vector is a possible answer is $\mathsf{A}^{\intercal}\boldsymbol{y} = \boldsymbol{c}$, where $\boldsymbol{c} = (0,\quad 0,\quad 2,\quad 2,)$.

□

(Final May 21/22) Exercise 1(c) From the elimination in part (a) we deduce that a basis of $\mathcal{N}(\mathbf{A})$ is $\left[\begin{pmatrix} 5 \\ -2 \\ -1 \\ 1 \end{pmatrix};\right]$.

□

(Final May 21/22) Exercise 1(d) Such orthogonal complement is the set of vectors orthogonal to $(5, \quad -2, \quad -1, \quad 1,)$, i.e.

$$\left\{ \boldsymbol{v} \in \mathbb{R}^4 \;\middle|\; \begin{bmatrix} 5 & -2 & -1 & 1 \end{bmatrix} \boldsymbol{v} = (0,) \right\}.$$

□

(Final May 21/22) Exercise 1(e) Obviously some Cartesian equations of $\mathcal{N}(\mathbf{A})$ are

$$\left\{ \boldsymbol{x} \in \mathbb{R}^4 \;\middle|\; \mathbf{A}\boldsymbol{x} = \mathbf{0} \right\}, \text{ es decir } \left\{ \boldsymbol{x} \in \mathbb{R}^4 \;\middle|\; \begin{bmatrix} 1 & 2 & 1 & 0 \\ 2 & 5 & 1 & 1 \\ 0 & 1 & 1 & 3 \end{bmatrix} \boldsymbol{x} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right\} = \mathcal{N}(\mathbf{A}).$$

□

(Final May 21/22) Exercise 2(a) Since $\mathbf{R}$ is 3×3, $\mathbf{A}$ must have 3 columns so that $\boxed{n = 3}$. The column space of $\mathbf{A}$ is spanned by 3 orthonormal vectors, so $\boxed{\text{the rank of } \mathbf{A} \text{ is } 3}$. The number of rows must be greater than or equal to the rank, so $\boxed{m \geq 3}$.

□

(Final May 21/22) Exercise 2(b) $\mathbf{A}_{|3} = \mathbf{Q}\mathbf{R}_{|3} = \mathbf{Q}\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$, that is, $\mathbf{A}_{|3} = 0\mathbf{Q}_{|1} + 1\mathbf{Q}_{|2} + 1\mathbf{Q}_{|3}$.

□

(Final May 21/22) Exercise 2(c) Since the columns of Q are *orthonormal*

$$\|\mathbf{A}_{|3}\|^2 = (\mathbf{Q}_{|2} + \mathbf{Q}_{|3}) \cdot (\mathbf{Q}_{|2} + \mathbf{Q}_{|3}) = (\mathbf{Q}_{|2}) \cdot (\mathbf{Q}_{|2}) + (\mathbf{Q}_{|3}) \cdot (\mathbf{Q}_{|3}) = 1 + 1 = 2.$$

Therefore $\|\mathbf{A}_{|3}\| = \sqrt{2}$.

□

(Final May 21/22) Exercise 2(d) Since $\mathbf{A}_{|1} = \mathbf{Q}_{|1}$, since $\mathbf{A}_{|3} = \mathbf{Q}_{|2} + \mathbf{Q}_{|3}$, and since columns of $\mathbf{Q}$ are orthogonal, then we know that the first and third columns of $\mathbf{A}$ are perpendicular.

$$(\mathbf{A}_{|1}) \cdot (\mathbf{A}_{|3}) = (\mathbf{Q}_{|1}) \cdot (\mathbf{Q}_{|2} + \mathbf{Q}_{|3}) = (\mathbf{Q}_{|1}) \cdot (\mathbf{Q}_{|2}) + (\mathbf{Q}_{|1}) \cdot (\mathbf{Q}_{|3}) = 0 + 0 = 0 \;\Rightarrow \mathbf{A}_{|1} \perp \mathbf{A}_{|3}.$$

However, all other dot products between columns of $\mathbf{A}$ are non-zero; e.g.

$$(\mathbf{A}_{|2}) \cdot (\mathbf{A}_{|3}) = (-3\mathbf{Q}_{|1} + 2\mathbf{Q}_{|2}) \cdot (\mathbf{Q}_{|2} + \mathbf{Q}_{|3}) = 2(\mathbf{Q}_{|2}) \cdot (\mathbf{Q}_{|2}) = 2.$$

□

(Final May 21/22) Exercise 2(e) If $\mathbf{A}$ is square, so it is $\mathbf{Q}$. Therefore $\mathbf{Q}$ is an orthogonal matrix, that is to say $\mathbf{Q}^\intercal\mathbf{Q} = \mathbf{I}$. Consequently $\det(\mathbf{Q}^\intercal\mathbf{Q}) = (\det \mathbf{Q})^2 = 1$. Therefore $\det \mathbf{Q}$ can only be 1 or -1. Thus,

$$|\det \mathbf{A}| = |\det(\mathbf{Q}\mathbf{R})| = |\det \mathbf{Q} \cdot \det \mathbf{R}| = |\det \mathbf{R}| = 2.$$

□

(Final May 21/22) Exercise 3(a)

$$\left[\begin{array}{ccc} 1 & 0 & 0 \\ a & 1 & 0 \\ 0 & a & 1 \\ \hline 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right] \xrightarrow[{[(-a)\mathbf{2}+\mathbf{1}]}]{\boldsymbol{\tau}} \left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -a^2 & a & 1 \\ \hline 1 & 0 & 0 \\ -a & 1 & 0 \\ 0 & 0 & 1 \end{array}\right] \xrightarrow[{\substack{[(a^2)\mathbf{3}+\mathbf{1}] \\ [(-a)\mathbf{3}+\mathbf{2}]}}]{\boldsymbol{\tau}} \left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ \hline 1 & 0 & 0 \\ -a & 1 & 0 \\ a^2 & -a & 1 \end{array}\right] \Rightarrow \mathbf{L}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -a & 1 & 0 \\ a^2 & -a & 1 \end{bmatrix}.$$

□

(Final May 21/22) Exercise 3(b) Note that $\mathbf{L}$ is invertible no matter what a is, and $\mathbf{D}$ is invertible so long as $d \neq 0$. So $\mathbf{A} = \mathbf{L}\mathbf{D}\mathbf{L}^\intercal$ will be invertible whenever $d \neq 0$. If $d = 0$, then of course $\mathbf{A}$ can't be invertible.

□

(Final May 21/22) Exercise 3(c) The matrix A is always symmetric, since $\mathbf{A}^\intercal = \left(\mathbf{L}\mathbf{D}\mathbf{L}^\intercal\right)^\intercal = \left(\mathbf{L}^\intercal\right)^\intercal \mathbf{D}^\intercal \mathbf{L}^\intercal = \mathbf{L}\mathbf{D}\mathbf{L}^\intercal = \mathbf{A}$.

□

(Final May 21/22) Exercise 3(d) $\mathbf{A}$ is positive definite if and only if $\boldsymbol{x}\mathbf{A}\boldsymbol{x} > 0$ for all $\boldsymbol{x} \neq 0$. If we call $\boldsymbol{y}$ the vector $\mathbf{L}^\intercal\boldsymbol{x}$, we have that

$$\boldsymbol{x}\mathbf{A}\boldsymbol{x} = \boldsymbol{x}\mathbf{L}\mathbf{D}\mathbf{L}^\intercal\boldsymbol{x} = \boldsymbol{y}\mathbf{D}\boldsymbol{y} = \boldsymbol{y}\begin{bmatrix} d & 0 & 0 \\ 0 & d^2 & 0 \\ 0 & 0 & d^3 \end{bmatrix}\boldsymbol{y},$$

which is greater than zero if and only if $d > 0$.

□

(Final May 21/22) Short questions set 1(a) Since $\mathbf{A}$ and $\mathbf{D}$ are similar, they have the same eigenvalues. And since $\mathbf{A}$ is diagonalizable, then $\mathbf{A}^k = \mathbf{X}(\mathbf{D}^k)(\mathbf{X}^{-1})$. Therefore

$$\mathbf{M} \;=\; \mathbf{A}^4 - 2(\mathbf{A}^2) - 8\mathbf{I} \;=\; \mathbf{X}\big(\mathbf{D}^4 - 2(\mathbf{D}^2) - 8\mathbf{I}\big)\mathbf{X}^{-1} \;=\; \mathbf{X}\begin{bmatrix} -9 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -9 \end{bmatrix}(\mathbf{X}^{-1}).$$

Thus, the eigenvalues are -9 (double) and 0 (double).

□

(Final May 21/22) Short questions set 1(b) The nonzero solutions of $\mathbf{M}\boldsymbol{x} = \mathbf{0}$ are the eigenvectors corresponding to $\lambda = 0$. Therefore, a basis of the corresponding eigenspace is formed by the columns 2 and 3 of $\mathbf{X}$, so that

$$\left\{ \boldsymbol{v} \in \mathbb{R}^4 \;\middle|\; \exists \boldsymbol{p} \in \mathbb{R}^2,\ \boldsymbol{v} = \begin{bmatrix} 1 & -1 \\ 1 & 2 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}\boldsymbol{p} \right\}.$$

□

(Final May 21/22) Short questions set 2(a) **True:** We know that $(\mathbf{A}^{-1})\mathbf{A} = \mathbf{I}$, and substituting $\mathbf{A} = \mathbf{A}^\intercal$ we have $(\mathbf{A}^{-1})(\mathbf{A}^\intercal) = \mathbf{I}$; by transposing we have that $\mathbf{A}(\mathbf{A}^{-1})^\intercal = \mathbf{I}$, that is, $(\mathbf{A}^{-1})^\intercal = \mathbf{A}^{-1}$.

□

(Final May 21/22) Short questions set 2(b) **False:** The rank of $\mathbf{Q}$ is n, since its n columns are linearly independent because they are perpendicular to each other. Therefore the m rows of $\mathbf{Q}$ are linearly dependent (since $m > \operatorname{rg}\big(\mathbf{Q}\big)$; that is, there is $\boldsymbol{y} \neq \mathbf{0}$ in $\mathbb{R}^m$ such that $\boldsymbol{y}\mathbf{Q} = \mathbf{0}$. Therefore $\mathbf{Q}(\mathbf{Q}^\intercal)\boldsymbol{y} = \mathbf{Q}\mathbf{0} = \mathbf{0}$, i.e., the columns of $\mathbf{Q}(\mathbf{Q}^\intercal)$ are linearly dependent (i.e., the square matrix $\mathbf{Q}(\mathbf{Q}^\intercal)$ is singular).

□

(Final May 21/22) Short questions set 2(c) **True:** If $\lambda = 0$, then $\mathbf{A} - 0\mathbf{I}$ is singular, i.e., $\mathbf{A}$ is singular. Therefore the columns of $\mathbf{A}$ are linearly dependent, i.e., there exists $\boldsymbol{x} \neq \mathbf{0}$ such that $\mathbf{A}\boldsymbol{x} = \mathbf{0}$.

□

(Final May 21/22) Short questions set 2(d) **True:** If $\mathbf{A}$ is symmetric, it is orthogonally diagonalizable: $\mathbf{A} = \mathbf{Q}^\intercal\mathbf{D}\mathbf{Q}$; Therefore $\mathbf{A}^2 = \mathbf{Q}^\intercal(\mathbf{D}^2)\mathbf{Q}$ is an orthogonal diagonalization of $\mathbf{A}^2$ where the elements of the diagonal of $\mathbf{D}^2$ are the eigenvalues of $\mathbf{A}^2$; which are necessarily all positive since they are the square of the eigenvalues of $\mathbf{A}$ (all nonzero since $\mathbf{A}$ is invertible).

□

(Final May 21/22) Short questions set 2(e) **False:** For example, $\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$.

□

(Final May 21/22) Short questions set 2(f) **False:** For example, $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

$\square$

(Final May 21/22) Short questions set 2(g) **False:**. It is possible to find linear combinations of $\mathbf{0}$ other than the trivial one, $0(\mathbf{0})$, that are the null vector. For example: $1492(\mathbf{0}) = \mathbf{0}$. Therefore, this set is linearly dependent.

$\square$

(Final May 21/22) Short questions set 3(a) Diagonalizing by congruence:

$$\begin{bmatrix} -1 & 0 & a \\ 0 & 0 & 0 \\ a & 0 & -4 \end{bmatrix} \xrightarrow[{[(a)\overset{\tau}{\mathbf{1}}+\mathbf{3}]}]{[(a)\overset{\tau}{\mathbf{1}}+\mathbf{3}]} \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & (a-2)(a+2) \end{bmatrix} \xrightarrow[{[\overset{\tau}{\mathbf{2}\rightleftharpoons\mathbf{3}}]}]{[\overset{\tau}{\mathbf{2}\rightleftharpoons\mathbf{3}}]} \begin{bmatrix} -1 & 0 & 0 \\ 0 & (a-2)(a+2) & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

$$\text{Quadratic form } f(x,y,z) \begin{cases} \text{is negative semi-definite if} & |a| \leq 2 \\ \text{is neither positive nor negative definite if} & |a| > 2 \end{cases}.$$

$\square$

(Final July 20/21) Exercise 1(a)

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & -1 \\ 2 & 0 & 0 & 0 & -2 \\ 0 & 3 & 3 & 0 & 0 \\ 0 & 3 & 4 & 1 & -3 \\ 0 & 1 & 2 & 1 & -3 \\ \hline 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{\substack{[(1)\overset{\tau}{\mathbf{1}}+\mathbf{5}] \\ [(-1)\mathbf{2}+\mathbf{3}]}} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 \\ 0 & 3 & 1 & 1 & -3 \\ 0 & 1 & 1 & 1 & -3 \\ \hline 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{\substack{[(-1)\overset{\tau}{\mathbf{3}}+\mathbf{4}] \\ [(3)\mathbf{3}+\mathbf{5}]}} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 \\ 0 & 3 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ \hline 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & -1 & 1 & -3 \\ 0 & 0 & 1 & -1 & 3 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right];$$

therefore, the solution set is $\left\{ \boldsymbol{v} \in \mathbb{R}^4 \;\middle|\; \exists \boldsymbol{p} \in \mathbb{R}^1,\ \boldsymbol{v} = \begin{pmatrix} 1 \\ -3 \\ 3 \\ 0 \end{pmatrix} + \begin{bmatrix} 0 \\ 1 \\ -1 \\ 1 \end{bmatrix} \boldsymbol{p} \right\}$. $\square$

(Final July 20/21) Exercise 1(b) It is a line, since $\dim \mathcal{N}\left(\mathbf{A}\right) = 1$. $\square$

(Final July 20/21) Exercise 1(c) For example, ${}_{1|}\mathbf{A}$, ${}_{3|}\mathbf{A}$, and ${}_{4|}\mathbf{A}$:

$$\text{a basis for } \mathcal{C}\left(\mathbf{A}^\intercal\right) : \left[\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}; \begin{pmatrix} 0 \\ 3 \\ 3 \\ 0 \end{pmatrix}; \begin{pmatrix} 0 \\ 3 \\ 4 \\ 1 \end{pmatrix}; \right].$$

$\square$

(Final July 20/21) Exercise 1(d) Solutions to $\mathbf{A}\boldsymbol{x} = \mathbf{0}$ are perpendicular to the rows of $\mathbf{A}$, hence,

$$\text{a basis for } \mathcal{C}\left(\mathbf{A}^\intercal\right)^\perp \left(\text{is basis for } \mathcal{N}\left(\mathbf{A}\right)\right) : \left[\begin{pmatrix} 0 \\ 1 \\ -1 \\ 1 \end{pmatrix}; \right].$$

$\square$

(Final July 20/21) Exercise 2(a) The eigenvalues are -1, 0, and 1, since $\mathbf{A}$ is triangular.

$$\text{for } \lambda = -1: \quad \left[\frac{\mathbf{A} - \lambda\mathbf{I}}{\mathbf{I}}\right] = \left[\begin{array}{ccc} 0 & 2 & 4 \\ 0 & 1 & 5 \\ 0 & 0 & 2 \\ \hline 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right] \xrightarrow{[(-2)\overset{\tau}{\mathbf{2}}+\mathbf{3}]} \left[\begin{array}{ccc} 0 & 2 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & 2 \\ \hline 1 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{array}\right]; \quad \mathcal{E}_{-1} = \mathcal{L}\left(\left[\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}; \right] \right)$$

$$\text{for } \lambda = 0: \quad \left[\frac{\mathbf{A} - \lambda\mathbf{I}}{\mathbf{I}}\right] = \left[\begin{array}{ccc} -1 & 2 & 4 \\ 0 & 0 & 5 \\ 0 & 0 & 1 \\ \hline 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right] \xrightarrow{\substack{[(2)\overset{\tau}{\mathbf{1}}+\mathbf{2}] \\ [(4)\mathbf{1}+\mathbf{3}]}} \left[\begin{array}{ccc} -1 & 0 & 0 \\ 0 & 0 & 5 \\ 0 & 0 & 1 \\ \hline 1 & 2 & 4 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right]; \quad \mathcal{E}_{0} = \mathcal{L}\left(\left[\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}; \right] \right)$$

for $\lambda = 1$: $\left[\begin{array}{c} \mathbf{A}-\lambda\mathbf{I} \\ \hline \mathbf{I} \end{array}\right] = \left[\begin{array}{ccc} -2 & 2 & 4 \\ 0 & -1 & 5 \\ 0 & 0 & 0 \\ \hline 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right] \xrightarrow{\substack{\tau \\ [(1)\mathbf{1}+\mathbf{2}] \\ [(2)\mathbf{1}+\mathbf{3}] \\ [(5)\mathbf{2}+\mathbf{3}]}} \left[\begin{array}{ccc} -2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \\ \hline 1 & 1 & 7 \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{array}\right]$; $\mathcal{E}_1 = \mathcal{L}\left(\left[\begin{pmatrix} 7 \\ 5 \\ 1 \end{pmatrix};\right]\right)$

Hence, $\mathbf{D} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $\mathbf{S} = \begin{bmatrix} 1 & 2 & 7 \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{bmatrix}$. $\square$

(Final July 20/21) Exercise 2(b) Since $\mathbf{D}^{1001} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{1001} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \mathbf{D}$ and $\mathbf{A} = \mathbf{SDS}^{-1}$ then

$$\boxed{\mathbf{A}^{1001} = \mathbf{SD}^{1001}\mathbf{S}^{-1} = \mathbf{SDS}^{-1} = \mathbf{A}}.$$

Since $\mathbf{I}$ is full rank but $\mathbf{A}^{1000}$ can't be full rank (since $\mathbf{A}$ is singular), then $\mathbf{A}^{1000} \neq \mathbf{I}$. $\square$

(Final July 20/21) Exercise 2(c) $\mathbf{A}^\intercal\mathbf{A}$ has 2 positive eigenvalues (it has rank 2, its eigenvalues can never be negative since $\boldsymbol{x}\mathbf{A}^\intercal\mathbf{A}\boldsymbol{x} = \mathbf{A}\boldsymbol{x}\cdot\mathbf{A}\boldsymbol{x} = \sum_i \left(\mathbf{A}\boldsymbol{x}_{|i}\right)^2 \geq 0$). One eigenvalue is zero because $\mathbf{A}^\intercal\mathbf{A}$ is singular.

(Or: $\mathbf{A}^\intercal\mathbf{A}$ is symmetric, so the eigenvalues have the same signs as the entries in the main diagonal after diagonalization by congruence:

$$\begin{bmatrix} 1 & -2 & -4 \\ -2 & 4 & 8 \\ -4 & 8 & 42 \end{bmatrix} \xrightarrow{\substack{\tau \\ [(2)\mathbf{1}+\mathbf{2}] \\ [(4)\mathbf{1}+\mathbf{3}]}} \begin{bmatrix} 1 & 0 & 0 \\ -2 & 0 & 0 \\ -4 & 0 & 26 \end{bmatrix} \xrightarrow[\substack{\tau \\ [(4)\mathbf{1}+\mathbf{3}] \\ [(2)\mathbf{1}+\mathbf{2}]}]{} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 26 \end{bmatrix} \xrightarrow{\substack{\tau \\ [\mathbf{2}\rightleftharpoons\mathbf{3}]}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 26 & 0 \end{bmatrix} \xrightarrow[\substack{\tau \\ [\mathbf{2}\rightleftharpoons\mathbf{3}]}]{} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 26 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\square$

(Final July 20/21) Exercise 2(d) No, for example: $\mathbf{A}^\intercal\mathbf{A}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \mathbf{A}^\intercal\begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ -4 \end{pmatrix}$. $\square$

(Final July 20/21) Exercise 3(a) $\mathbf{A}^{-1}$ has eigenvalues $\frac{1}{\lambda_j}$ with the same eigenvectors. Proof:

$$\mathbf{A}\boldsymbol{q}_j = \lambda_j\boldsymbol{q}_j \iff \left(\mathbf{A}^{-1}\right)\mathbf{A}\boldsymbol{q}_j = \left(\mathbf{A}^{-1}\right)\lambda_j\boldsymbol{q}_j \iff \boldsymbol{q}_j = \lambda_j\left(\mathbf{A}^{-1}\right)\boldsymbol{q}_j \iff \boxed{\left(\mathbf{A}^{-1}\right)\boldsymbol{q}_j = \frac{1}{\lambda_j}\boldsymbol{q}_j}.$$

$\square$

(Final July 20/21) Exercise 3(b) Multiply $\boldsymbol{b} = c_1\boldsymbol{q}_1 + \cdots + c_n\boldsymbol{q}_n$ by $\boldsymbol{q}_1$. Orthonormality gives

$$\boldsymbol{b}\cdot\boldsymbol{q}_1 = (c_1\boldsymbol{q}_1 + \cdots + c_n\boldsymbol{q}_n)\cdot\boldsymbol{q}_1 = c_1\boldsymbol{q}_1\cdot\boldsymbol{q}_1 = c_1 \quad \text{so} \quad \boxed{c_1 = \boldsymbol{b}\cdot\boldsymbol{q}_1}.$$

$\square$

(Final July 20/21) Exercise 3(c) Multiplying $\boldsymbol{b}$ by $\mathbf{A}^{-1}$ will multiply each $\boldsymbol{q}_j$ by $\frac{1}{\lambda_j}$ (part (a)).

$$\mathbf{A}^{-1}\boldsymbol{b} = \mathbf{A}^{-1}(c_1\boldsymbol{q}_1 + \cdots + c_n\boldsymbol{q}_n) = c_1\left(\mathbf{A}^{-1}\right)\boldsymbol{q}_1 + \cdots + c_n\left(\mathbf{A}^{-1}\right)\boldsymbol{q}_n = \frac{c_1}{\lambda_1}\boldsymbol{q}_1 + \cdots + \frac{c_n}{\lambda_n}\boldsymbol{q}_n.$$

So d_1 becomes

$$\boxed{d_1 = \frac{c_1}{\lambda_1}} \quad \text{or using part (b):} \quad d_1 = \frac{\boldsymbol{b}\cdot\boldsymbol{q}_1}{\lambda_1}.$$

$\square$

(Final July 20/21) Short questions set 1(a) Since $\mathbf{A}$ has three orthonormal columns : $\mathbf{A}^\intercal\mathbf{A} = \underset{3\times 3}{\mathbf{I}}$. $\square$

(Final July 20/21) Short questions set 1(b) Since $\operatorname{rg}(\mathbf{AB}) \leq \operatorname{rg}(\mathbf{B})$, then

$$\boxed{\operatorname{rg}\left(\mathbf{AA}^\intercal\right) \leq \operatorname{rg}\left(\mathbf{A}^\intercal\right) = \operatorname{rg}(\mathbf{A}) = 3}$$

(so $\mathbf{A}\mathbf{A}^{\mathsf{T}}$, of order 5, is singular).

In fact, $\boxed{\operatorname{rg}\left(\mathbf{A}\mathbf{A}^{\mathsf{T}}\right)=\operatorname{rg}\left(\mathbf{A}^{\mathsf{T}}\right)=3}$*, lets's see why: on the one hand,* $\mathbf{A}^{\mathsf{T}}$ *and* $\mathbf{A}\mathbf{A}^{\mathsf{T}}$ *have* 5 *columns, and on the other hand* $\dim\mathcal{N}\left(\mathbf{A}\mathbf{A}^{\mathsf{T}}\right)=\dim\mathcal{N}\left(\mathbf{A}^{\mathsf{T}}\right)=2$ *since both subspaces are equal: if* $\boldsymbol{x}\in\mathcal{N}\left(\mathbf{A}^{\mathsf{T}}\right)$ *then*

$$\mathbf{A}^{\mathsf{T}}\boldsymbol{x}=\mathbf{0}\Rightarrow\mathbf{A}\mathbf{A}^{\mathsf{T}}\boldsymbol{x}=\mathbf{0}\Rightarrow\boldsymbol{x}\in\mathcal{N}\left(\mathbf{A}\mathbf{A}^{\mathsf{T}}\right)$$

and if $\boldsymbol{x}\in\mathcal{N}\left(\mathbf{A}\mathbf{A}^{\mathsf{T}}\right)$ *then*

$$\mathbf{A}\mathbf{A}^{\mathsf{T}}\boldsymbol{x}=\mathbf{0}\Rightarrow\boldsymbol{x}\mathbf{A}\mathbf{A}^{\mathsf{T}}\boldsymbol{x}=\boldsymbol{x}\cdot\mathbf{0}=0\Rightarrow\|\mathbf{A}^{\mathsf{T}}\boldsymbol{x}\|^2=0\Rightarrow\mathbf{A}^{\mathsf{T}}\boldsymbol{x}=\mathbf{0}\Rightarrow\boldsymbol{x}\in\mathcal{N}\left(\mathbf{A}^{\mathsf{T}}\right).$$

$\square$

(Final July 20/21) Short questions set 1(c) $\det\left(\mathbf{A}\left(\mathbf{A}^{\mathsf{T}}\mathbf{A}\right)^{-1}\mathbf{A}^{\mathsf{T}}\right)=\det\mathbf{A}\mathbf{I}\mathbf{A}^{\mathsf{T}}=\det\mathbf{A}\mathbf{A}^{\mathsf{T}}=0$ since $\mathbf{A}\mathbf{A}^{\mathsf{T}}$, of order 5, is singular.

$\square$

(Final July 20/21) Short questions set 2(a) False: Neither $\boldsymbol{u}$ nor $\boldsymbol{w}$ are solutions of these Cartesian equations; therefore those Cartesian equations do not correspond to subspace corresponding to subspace $\mathcal{V}=\mathcal{L}(\boldsymbol{u},\boldsymbol{w})$.

$\square$

(Final July 20/21) Short questions set 2(b) True: $\left(\left(\mathbf{A}^{\mathsf{T}}\mathbf{A}\right)^{-1}\mathbf{A}^{\mathsf{T}}\right)\mathbf{A}=\left(\mathbf{A}^{\mathsf{T}}\mathbf{A}\right)^{-1}\mathbf{A}^{\mathsf{T}}\mathbf{A}=\mathbf{I}$.

$\square$

(Final July 20/21) Short questions set 2(c) True: Eigenvectors of a matrix are also eigenvector of its inverse. Hence, if $\mathbf{A}\boldsymbol{v}=\lambda\boldsymbol{v}$ and $\mathbf{B}\boldsymbol{v}=\gamma\boldsymbol{v}$, then: $\mathbf{A}\mathbf{B}^{-1}\boldsymbol{v}=\mathbf{A}\left(\frac{1}{\gamma}\boldsymbol{v}\right)=\frac{1}{\gamma}\mathbf{A}\boldsymbol{v}=\frac{\lambda}{\gamma}\boldsymbol{v}$.

$\square$

(Final July 20/21) Short questions set 2(d) False. The set is not close under scalar multiplication. For example: $\boldsymbol{x}=\frac{1}{2}\begin{pmatrix}1\\1\\1\end{pmatrix}$ is not in the set.

$\square$

(Final July 20/21) Short questions set 3(a) It is **indefinite** since $q(x,y)=\boldsymbol{x}\mathbf{A}\boldsymbol{x}$, where $\mathbf{A}=\begin{bmatrix}0&2\\2&0\end{bmatrix}$, and

$$\mathbf{A}=\begin{bmatrix}0&2\\2&0\end{bmatrix}\xrightarrow{[(1)\overset{\boldsymbol{\tau}}{\mathbf{2}}+\mathbf{1}]}\begin{bmatrix}2&2\\2&0\end{bmatrix}\xrightarrow[{[(1)\overset{\boldsymbol{\tau}}{\mathbf{2}}+\mathbf{1}]}]{}\begin{bmatrix}4&2\\2&0\end{bmatrix}\xrightarrow{\substack{[(\overset{\boldsymbol{\tau}}{2})\mathbf{2}]\\ [(-1)\mathbf{1}+\mathbf{2}]}}\begin{bmatrix}4&0\\2&-2\end{bmatrix}\xrightarrow[{\substack{[(-1)\overset{\boldsymbol{\tau}}{\mathbf{1}}+\mathbf{2}]\\ [(2)\mathbf{2}]}}]{}\begin{bmatrix}4&0\\0&-4\end{bmatrix}=\mathbf{D}.$$

$\square$

(Final July 20/21) Short questions set 3(b) We have already seen that

$$\mathbf{D}=\underset{[(-1)\overset{\boldsymbol{\tau}}{\mathbf{1}}+\mathbf{2}][(\overset{\boldsymbol{\tau}}{2})\mathbf{2}][(1)\overset{\boldsymbol{\tau}}{\mathbf{2}}+\mathbf{1}]}{}\mathbf{A}_{[(1)\overset{\boldsymbol{\tau}}{\mathbf{2}}+\mathbf{1}][(\overset{\boldsymbol{\tau}}{2})\mathbf{2}][(-1)\overset{\boldsymbol{\tau}}{\mathbf{1}}+\mathbf{2}]}=\mathbf{B}^{\mathsf{T}}\mathbf{A}\mathbf{B},\qquad\text{where }\mathbf{B}=\mathbf{I}_{[(1)\overset{\boldsymbol{\tau}}{\mathbf{2}}+\mathbf{1}][(\overset{\boldsymbol{\tau}}{2})\mathbf{2}][(-1)\overset{\boldsymbol{\tau}}{\mathbf{1}}+\mathbf{2}]}.$$

Applying the inverse transformations to the columns of $\mathbf{I}$ we get

$$\mathbf{I}=\begin{bmatrix}1&0\\0&1\end{bmatrix}\xrightarrow{\substack{[(1)\overset{\boldsymbol{\tau}}{\mathbf{1}}+\mathbf{2}]\\ [(\frac{1}{2})\mathbf{2}]}}\begin{bmatrix}1&\frac{1}{2}\\0&\frac{1}{2}\end{bmatrix}\xrightarrow{[(-1)\overset{\boldsymbol{\tau}}{\mathbf{2}}+\mathbf{1}]}\begin{bmatrix}\frac{1}{2}&\frac{1}{2}\\-\frac{1}{2}&\frac{1}{2}\end{bmatrix}=\mathbf{B}^{-1};$$

since $\mathbf{B}^{-1}\boldsymbol{x}=\begin{bmatrix}\frac{1}{2}&\frac{1}{2}\\-\frac{1}{2}&\frac{1}{2}\end{bmatrix}\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}\frac{x}{2}+\frac{y}{2}\\-\frac{x}{2}+\frac{y}{2}\end{pmatrix}$; and since$\mathbf{A}=\left(\mathbf{B}^{-1}\right)^{\mathsf{T}}\mathbf{D}\mathbf{B}^{-1}$, we conclude that

$$q(x,y)=\boldsymbol{x}\mathbf{A}\boldsymbol{x}=\boldsymbol{x}\left(\mathbf{B}^{-1}\right)^{\mathsf{T}}\mathbf{D}\mathbf{B}^{-1}\boldsymbol{x}=\left(\tfrac{x}{2}+\tfrac{y}{2},\quad-\tfrac{x}{2}+\tfrac{y}{2},\right)\begin{bmatrix}4&0\\0&-4\end{bmatrix}\begin{pmatrix}\frac{x}{2}+\frac{y}{2}\\-\frac{x}{2}+\frac{y}{2}\end{pmatrix}=\boxed{4\left(\tfrac{x}{2}+\tfrac{y}{2}\right)^2-4\left(-\tfrac{x}{2}+\tfrac{y}{2}\right)^2}.$$

$\square$

(Final June 20/21) Exercise 1(a) Since $\mathbf{A}$ is symmetric, it is diagonalizable for any a. In the case of $\mathbf{B}$ we must find the set of values of a that makes the eigenspace corresponding to $\lambda = 1$ have dimension 2.

$$\mathbf{B}-\mathbf{I} = \begin{bmatrix} 0 & a & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{\overset{\boldsymbol{\tau}}{[(-\frac{1}{2})\mathbf{3}+\mathbf{2}]}} \begin{bmatrix} 0 & a-1 & 2 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}.$$

Hence, only when $a = 1$ the dimension of $\mathcal{N}\left(\mathbf{B}-\mathbf{I}\right)$ is 2.

□

(Final June 20/21) Exercise 1(b) Matrices $\mathbf{BAB}^{\text{-}1}$ and $\mathbf{A}$ are <u>*similar*</u> so they have the same trace: $\boxed{\operatorname{tr}\left(\mathbf{BAB}^{\text{-}1}\right) = 2+a}$. On the other hand, since $\mathbf{A}$ is singular ($\mathbf{A}_{|1} = \mathbf{A}_{|3}$), so it is $\mathbf{AB}^2$ and therefore $\boxed{|\mathbf{AB}^2| = 0}$.

□

(Final June 20/21) Exercise 1(c)

$$\begin{bmatrix} 1 & -1 & 1 \\ -1 & a & -1 \\ 1 & -1 & 1 \end{bmatrix} \xrightarrow{\substack{\overset{\boldsymbol{\tau}}{[(1)\mathbf{1}+\mathbf{2}]} \\ [(-1)\mathbf{1}+\mathbf{3}]}} \begin{bmatrix} 1 & 0 & 0 \\ -1 & a-1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \xrightarrow[\substack{\overset{\boldsymbol{\tau}}{[(-1)\mathbf{1}+\mathbf{3}]} \\ [(1)\mathbf{1}+\mathbf{2}]}]{} \begin{bmatrix} 1 & 0 & 0 \\ 0 & a-1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{cases} a \geq 1 & \text{Positive semidefinite} \\ a < 1 & \text{Indefinite} \end{cases}$$

□

(Final June 20/21) Exercise 1(d)

for $\lambda = 2$: $$\left[\begin{array}{c} \mathbf{B}-2\mathbf{I} \\ \hline \mathbf{I} \end{array}\right] = \left[\begin{array}{ccc} -1 & 1 & 2 \\ 0 & 0 & 2 \\ 0 & 0 & -1 \\ \hline 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right] \xrightarrow{\substack{\overset{\boldsymbol{\tau}}{[(1)\mathbf{1}+\mathbf{2}]} \\ [(2)\mathbf{1}+\mathbf{3}]}} \left[\begin{array}{ccc} -1 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & -1 \\ \hline 1 & 1 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right]; \quad \mathcal{E}_2 = \mathcal{L}\left(\left[\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix};\right]\right)$$

for $\lambda = 1$: $$\left[\begin{array}{c} \mathbf{B}-\mathbf{I} \\ \hline \mathbf{I} \end{array}\right] = \left[\begin{array}{ccc} 0 & 1 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \\ \hline 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right] \xrightarrow{\overset{\boldsymbol{\tau}}{[(-2)\mathbf{2}+\mathbf{3}]}} \left[\begin{array}{ccc} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ \hline 1 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{array}\right]; \quad \mathcal{E}_1 = \mathcal{L}\left(\left[\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}; \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix};\right]\right)$$

$$\text{Hence, } \mathbf{D} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^3 = \begin{bmatrix} 8 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{S} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & -2 \\ 0 & 0 & 1 \end{bmatrix}.$$

□

(Final June 20/21) Exercise 1(e) Since eigenspaces $\mathcal{E}_1$ and $\mathcal{E}_2$ are not perpendicular, it is not posible to find and ortonormal basis formed by eigenvectors of $\mathbf{B}$.

□

(Final June 20/21) Exercise 2(a) Since $\boldsymbol{b}$ is a multiple of $(\mathbf{A}_5)_{|,}$ this system of equations is always solvable.

Since there are less rows than columns, the rank is less than the number of columns, hence, columns are linearly dependent, and therefore, *there are infinite solutions for this system regardless of the value of the parameters*.

□

(Final June 20/21) Exercise 2(b)

$$\left[\begin{array}{ccccc|c} 1 & a & 2 & 1 & 1 & -c \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ \hline 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{\substack{\overset{\boldsymbol{\tau}}{[(-1)\mathbf{5}+\mathbf{1}]} \\ [(-a)\mathbf{5}+\mathbf{2}] \\ [(-2)\mathbf{5}+\mathbf{3}] \\ [(-1)\mathbf{5}+\mathbf{4}] \\ [(c)\mathbf{5}+\mathbf{6}]}} \left[\begin{array}{ccccc|c} 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ \hline 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ -1 & -a & -2 & -1 & 1 & c \\ \hline 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{\substack{\overset{\boldsymbol{\tau}}{[(-1)\mathbf{4}+\mathbf{1}]} \\ [(-1)\mathbf{3}+\mathbf{2}]}} \left[\begin{array}{ccccc|c} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \hline 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 2-a & -2 & -1 & 1 & c \\ \hline 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right].$$

So the set of solutions is

$$\left\{ \boldsymbol{v}\in\mathbb{R}^5 \;\middle|\; \exists\boldsymbol{p}\in\mathbb{R}^2,\ \boldsymbol{v}=\begin{pmatrix}0\\0\\0\\0\\c\end{pmatrix}+\begin{bmatrix}1&0\\0&1\\0&-1\\-1&0\\0&2-a\end{bmatrix}\boldsymbol{p}\right\}.$$

□

(Final June 20/21) Exercise 2(c)

A) No, since in any solution $x_3=-x_2$.

B) Only when $a\neq 2$. Then we can divide the second special solution by $2-a$ and substract c times that vector from the particular solution; thus we get the following parametric equations:

$$\left\{ \boldsymbol{v}\in\mathbb{R}^5 \;\middle|\; \exists\boldsymbol{p}\in\mathbb{R}^2,\ \boldsymbol{v}=\begin{pmatrix}0\\\frac{c}{a-2}\\-\frac{c}{a-2}\\0\\0\end{pmatrix}+\begin{bmatrix}1&0\\0&-\frac{1}{a-2}\\0&\frac{1}{a-2}\\-1&0\\0&1\end{bmatrix}\boldsymbol{p}\right\}$$

□

(Final June 20/21) Exercise 2(d) Since the rank of $\mathbf{A}$ is 3 and the order of $\mathbf{A}^{\intercal}\mathbf{A}$ is 5, matrix $\mathbf{A}^{\intercal}\mathbf{A}$ is singular, and therefore its determinant is 0.

□

(Final June 20/21) Exercise 3(a) $\boxed{\boldsymbol{b}=\begin{pmatrix}0\\1\\0\end{pmatrix}}$; since $\boldsymbol{b}=\mathbf{A}\boldsymbol{c}=\mathbf{A}\mathbf{A}^{-1}{}_{|2}=(\mathbf{I}_2)_{|=}\begin{pmatrix}0\\1\\0\end{pmatrix}.$

□

(Final June 20/21) Exercise 3(b) $\boxed{\boldsymbol{d}=\begin{pmatrix}2\\-1\\0\end{pmatrix}}$; since $\boldsymbol{d}=\mathbf{UL}\boldsymbol{c}=\mathbf{UL}\left(\mathbf{A}^{-1}\right)_{|2}=\mathbf{UL}\left(\mathbf{L}^{-1}\mathbf{U}^{-1}\mathbf{B}\right)_{|2}=$ $(\mathbf{B}_2)_{|=}\begin{pmatrix}2\\-1\\0\end{pmatrix}.$

□

(Final June 20/21) Exercise 3(c) To get $\boldsymbol{c}$ we can just solve the system $\mathbf{UL}\boldsymbol{c}=\begin{pmatrix}2\\-1\\0\end{pmatrix}=\boldsymbol{d}$. But that system is equivalent to any system $\mathbf{EUL}\boldsymbol{c}=\mathbf{E}\boldsymbol{d}$ for any non-singular $\mathbf{E}$. Hence, we can multiply by $\mathbf{U}^{-1}$:

$$\left[\begin{array}{c}\mathbf{U}\\\hline\mathbf{I}\end{array}\right]=\left[\begin{array}{ccc}1&-3&7\\0&1&0\\0&0&1\\\hline 1&0&0\\0&1&0\\0&0&1\end{array}\right]\xrightarrow[{[(-7)\mathbf{1}+\mathbf{3}]}]{\overset{\boldsymbol{\tau}}{[(3)\mathbf{1}+\mathbf{2}]}}\left[\begin{array}{ccc}1&0&0\\0&1&0\\0&0&1\\\hline 1&3&-7\\0&1&0\\0&0&1\end{array}\right]=\left[\begin{array}{c}\mathbf{I}\\\hline\mathbf{U}^{-1}\end{array}\right]$$

Thus, $\boldsymbol{c}$ verifies

$$\mathbf{UL}\boldsymbol{c}=\boldsymbol{d}\quad\Rightarrow\quad\mathbf{L}\boldsymbol{c}=\mathbf{U}^{-1}\boldsymbol{d}=\begin{bmatrix}1&3&-7\\0&1&0\\0&0&1\end{bmatrix}\begin{pmatrix}2\\-1\\0\end{pmatrix}=\begin{pmatrix}-1\\-1\\0\end{pmatrix}=\boldsymbol{f};$$

so we can just solve $\mathbf{L}\boldsymbol{c}=\boldsymbol{f}$:

$$\left[\begin{array}{c|c}\mathbf{L}&-\boldsymbol{f}\\\hline\mathbf{I}&\mathbf{0}\end{array}\right]=\left[\begin{array}{ccc|c}1&0&0&1\\1&1&0&1\\5&-12&1&0\\\hline 1&0&0&0\\0&1&0&0\\0&0&1&0\\\hline 0&0&0&1\end{array}\right]\xrightarrow{\overset{\boldsymbol{\tau}}{[(-1)\mathbf{1}+\mathbf{4}]}}\left[\begin{array}{ccc|c}1&0&0&0\\1&1&0&0\\5&-12&1&-5\\\hline 1&0&0&-1\\0&1&0&0\\0&0&1&0\\\hline 0&0&0&1\end{array}\right]\xrightarrow{\overset{\boldsymbol{\tau}}{[(5)\mathbf{3}+\mathbf{4}]}}\left[\begin{array}{ccc|c}1&0&0&0\\1&1&0&0\\5&-12&1&0\\\hline 1&0&0&-1\\0&1&0&0\\0&0&1&5\\\hline 0&0&0&1\end{array}\right]$$

Hence, $\boldsymbol{c}$ (i.e., the second column of $\mathbf{A}^{-1}$) is $\begin{pmatrix} -1 \\ 0 \\ 5 \end{pmatrix}$.

□

(Final June 20/21) Short questions set 1(a) **The answer to the first question is NO**: since $\mathbf{A}\mathbf{A}^\intercal$ (of order 2) is full rank, then $\mathbf{A}^\intercal$ has 2 linealy independent columns (i.e., $\mathbf{A}^\intercal \boldsymbol{x} = \mathbf{0}$ if and only if $\boldsymbol{x} = \mathbf{0}$)

The answer to the second question is YES: The 5 columns of $\mathbf{A}$ are linearly dependent since the rank is only 2.

□

(Final June 20/21) Short questions set 1(b) Since the order of $\mathbf{A}\mathbf{A}^\intercal$ is 2, then $|2\left(\mathbf{A}\mathbf{A}^\intercal\right)^{-1}| = 2^2|\left(\mathbf{A}\mathbf{A}^\intercal\right)^{-1}| = \frac{4}{|\mathbf{A}\mathbf{A}^\intercal|} = \frac{4}{3}$.

□

(Final June 20/21) Short questions set 1(c) Since the order of $\mathbf{A}\mathbf{A}^\intercal = \left[\boldsymbol{c}_1;\ \boldsymbol{c}_2\right]$ is 2, then:

$$\det\left[\boldsymbol{c}_2;\ (3\boldsymbol{c}_1 + 2\boldsymbol{c}_2)\right] = \det\left[\boldsymbol{c}_2;\ 3\boldsymbol{c}_1\right] = 3\det\left[\boldsymbol{c}_2;\ \boldsymbol{c}_1\right] = -3\det\left[\boldsymbol{c}_1;\ \boldsymbol{c}_2\right] = -3|\mathbf{A}\mathbf{A}^\intercal| = -9.$$

□

(Final June 20/21) Short questions set 1(d) Since $\mathbf{A}\mathbf{A}^\intercal$ is 2 by 2 with rank 2, its columns are linearly independent; hence, $\boxed{\dim \mathcal{S} = 0}$.

□

(Final June 20/21) Short questions set 1(e) Since $\mathbf{A}^\intercal\mathbf{A}$ is 5 by 5 with rank 2, $\boxed{\dim \mathcal{W} = 5 - 2 = 3}$.

□

(Final June 20/21) Short questions set 1(f) Since the rank of $\mathbf{A}^\intercal\mathbf{A}$ is 2, $\boxed{\dim \mathcal{Z} = 2}$.

□

(Final June 20/21) Short questions set 1(g) We know dimension of $\mathcal{W}$ is 3; and $\mathcal{W}$ is the eigenspace for $\lambda = 0$. Hence, 3 is the geometric multiplicity for $\lambda = 0$.

Since $\mathbf{A}^\intercal\mathbf{A}$ is symetric, it is diagonalizable. It follows that geometric and algebraic multiplicities are equal, so 3 is also the algebraic multiplicity.

□

(Final June 20/21) Short questions set 1(h) Both, $\mathbf{A}\mathbf{A}^\intercal$ and $\mathbf{A}^\intercal$ have two columns, and both have rank two. So columns of $\mathbf{A}^\intercal$ are linearly independent, thus $\mathbf{A}^\intercal\boldsymbol{x} \neq \mathbf{0}$ when $\boldsymbol{x} \neq \mathbf{0}$.

If we denote $\mathbf{A}^\intercal\boldsymbol{x}$ with $\boldsymbol{v}$, it is clear that for any $\boldsymbol{x} \neq \mathbf{0}$

$$\boldsymbol{x}\mathbf{A}\mathbf{A}^\intercal\boldsymbol{x} = \boldsymbol{v}\cdot\boldsymbol{v} = \sum v_i^2 > 0 \quad \text{since} \quad \boldsymbol{v} \neq \mathbf{0} \quad \text{when} \quad \boldsymbol{x} \neq \mathbf{0}.$$

□

(Final June 20/21) Short questions set 2(a) For example $\boxed{P = \left\{\boldsymbol{v} \in \mathbb{R}^3 \;\middle|\; \begin{bmatrix} 0 & 1 & 1 \end{bmatrix}\boldsymbol{v} = (1,)\right\}}$ since

$$\left[\begin{array}{ccc} 1 & -1 & 1 \\ 2 & 1 & -1 \\ \hline x & y & z \\ \hline 1 & 0 & 1 \end{array}\right] \xrightarrow[{[(-1)\mathbf{1}+\mathbf{3}]}]{\substack{\boldsymbol{\tau} \\ [(1)\mathbf{1}+\mathbf{2}]}} \left[\begin{array}{ccc} 1 & 0 & 0 \\ 2 & 3 & -3 \\ \hline x & x+y & -x+z \\ \hline 1 & 1 & 0 \end{array}\right] \xrightarrow{\substack{\boldsymbol{\tau} \\ [(1)\mathbf{2}+\mathbf{3}]}} \left[\begin{array}{ccc} 1 & 0 & 0 \\ 2 & 3 & 0 \\ \hline x & x+y & y+z \\ \hline 1 & 1 & 1 \end{array}\right]$$

□

(Final June 20/21) Short questions set 2(b) Vector $\boldsymbol{c}$, since it is the only one that satisfies the Cartesian equations.

□

(Final June 18/19) Exercise 1(a) $\mathbf{C}_{|3} = \left(\mathbf{A}\mathbf{B}\right)_{|3} = \mathbf{A}\left(\mathbf{B}_{|3}\right) = \mathbf{A}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \mathbf{A}_{|1}$.

□

(Final June 18/19) Exercise 1(b) On the one hand, since

$$\underset{3\times5}{\mathbf{C}} = \underset{m\times n}{\mathbf{A}}\ \underset{3\times5}{\mathbf{B}}$$

then $\mathbf{A}$ has three columns ($n=3$); and $\mathbf{A}$ has as many rows as $\mathbf{C}$ ($m=3$); so $\mathbf{A}$ is a 3 by 3 matrix. On the other hand, since $\mathbf{C}$ has rank three, then the rank of $\mathbf{A}$ can not be less than 3; Hence $\boxed{\mathbf{A} \text{ is invertible}}$.

Although not required, we can find the inverse of $\mathbf{A}$.

By columns: from the first three columns we can see that

$$\left.\begin{array}{rl} \mathbf{A}\mathbf{B}_{|3} = & 2\mathbf{I}_{|1} \\ \mathbf{A}\mathbf{B}_{|2} = & \mathbf{I}_{|2} \\ \mathbf{A}\mathbf{B}_{|1} = & \mathbf{I}_{|3} \end{array}\right\} \Rightarrow \mathbf{A}\left[\tfrac{1}{2}\mathbf{B}_{|3} \quad \mathbf{B}_{|2} \quad \mathbf{B}_{|1}\right] = \underset{3\times 3}{\mathbf{I}}. \text{ So } \mathbf{A}^{-1} = \left[\tfrac{1}{2}\mathbf{B}_{|3} \quad \mathbf{B}_{|2} \quad \mathbf{B}_{|1}\right] = \begin{bmatrix} \frac{1}{2} & -1 & 1 \\ 0 & 6 & 0 \\ 0 & 4 & 2 \end{bmatrix}.$$

By rows: Since $\mathbf{AB} = \mathbf{C}$, we known that $\mathbf{A}^{-1}\mathbf{C} = \mathbf{B}$. Therefore, if we start from $\left[\mathbf{C}\,|\,\mathbf{I}\right]$ and if, by elementary row operations, we transform $\mathbf{C}$ in $\mathbf{B}$, then $\mathbf{I}$ becomes $\mathbf{A}^{-1}$.

$$\left[\begin{array}{ccccc|ccc} 0 & 0 & 2 & 2 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 3 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow[{[(\frac{1}{2})\mathbf{1}]}]{\tau} \left[\begin{array}{ccccc|ccc} 0 & 0 & 1 & 1 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 3 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow[{[(1)\mathbf{3}+\mathbf{1}]}]{\tau}$$

$$\left[\begin{array}{ccccc|ccc} 1 & 0 & 1 & 4 & 1 & \frac{1}{2} & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 3 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow[{[(-1)\mathbf{2}+\mathbf{1}]}]{\tau} \left[\begin{array}{ccccc|ccc} 1 & -1 & 1 & 4 & 0 & \frac{1}{2} & -1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 3 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow[{[(2)\mathbf{3}]}]{\tau}$$

$$\left[\begin{array}{ccccc|ccc} 1 & -1 & 1 & 4 & 0 & \frac{1}{2} & -1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 2 & 0 & 0 & 6 & 2 & 0 & 0 & 2 \end{array}\right] \xrightarrow[{[(4)\mathbf{2}+\mathbf{3}]}]{\tau} \left[\begin{array}{ccccc|ccc} 1 & -1 & 1 & 4 & 0 & \frac{1}{2} & -1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 2 & 4 & 0 & 6 & 6 & 0 & 4 & 2 \end{array}\right] \xrightarrow[{[(6)\mathbf{2}]}]{\tau}$$

$$\left[\begin{array}{ccccc|ccc} 1 & -1 & 1 & 4 & 0 & \frac{1}{2} & -1 & 1 \\ 0 & 6 & 0 & 0 & 6 & 0 & 6 & 0 \\ 2 & 4 & 0 & 6 & 6 & 0 & 4 & 2 \end{array}\right] = \left[\mathbf{B}|\mathbf{A}^{-1}\right].$$

□

(Final June 18/19) Exercise 1(c) Since $\mathbf{A}$ is invertible, then $\mathbf{A}\boldsymbol{x} = \mathbf{C}_{|5}$ implies $\boldsymbol{x} = \mathbf{A}^{-1}\mathbf{C}_{|5}$. But, since $\mathbf{B} = \mathbf{A}^{-1}\mathbf{C}$, then $\boxed{\boldsymbol{x} = \left(\mathbf{A}^{-1}\mathbf{C}\right)_{|5} = \mathbf{B}_{|5} = \begin{pmatrix} 0 \\ 6 \\ 6 \end{pmatrix}}$ (the fith column of $\mathbf{B}$).

If you already known $\mathbf{A}^{-1}$, you can also compute: $\boldsymbol{x} = \mathbf{A}^{-1}\mathbf{C}_{|5} = \begin{bmatrix} \frac{1}{2} & -1 & 1 \\ 0 & 6 & 0 \\ 0 & 4 & 2 \end{bmatrix} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 6 \\ 6 \end{pmatrix}$.

□

(Final June 18/19) Exercise 1(d) **Yes**. Since the rank of $\mathbf{B}$ is three, then $\dim \mathcal{C}\left(\mathbf{B}^{\mathsf{T}}\right) = 3$. And since $\mathbf{C} = \mathbf{AB}$, rows of $\mathbf{C}$ are linear combinations of rows of $\mathbf{B}$, but rows of $\mathbf{C}$ are linearly independent, since $\operatorname{rg}\left(\mathbf{C}\right) = 3$. Therefore, rows of $\mathbf{C}$ are a basis for $\mathcal{C}\left(\mathbf{B}^{\mathsf{T}}\right)$.

□

(Final June 18/19) Exercise 1(e) Note that

$$\mathbf{A}\left(\mathbf{B}_{|3}\right) = 2\cdot\left(\mathbf{B}_{|3}\right); \qquad \mathbf{A}\left(\mathbf{B}_{|4}\right) = \frac{1}{2}\cdot\left(\mathbf{B}_{|4}\right); \qquad \mathbf{A}\left(\mathbf{B}_{|5}\right) = \frac{1}{6}\cdot\left(\mathbf{B}_{|5}\right);$$

Hence, the three last columns of $\mathbf{B}$ are eigenvectors of $\mathbf{A}$, corresponding respectively to the eigenvalues 2, $\frac{1}{2}$, y $\frac{1}{6}$. Since each eigenvector correspond to a different eigenvalue, they are linearly independent and form a basis for $\mathbb{R}^3$.

□

(Final June 18/19) Exercise 2(a) Puesto que la matriz es de orden 3 y de rango 2 (pues dos autovalores son distintos de cero y solo uno igual cero), el conjunto de soluciones son todos los múltiplos del autovector asociado al autovalor 0; es decirSince the 3 by 3 matrix has rank 2 (only two eigenvalues are nonzero) then $\dim \mathcal{N}\left(\mathbf{A}\right) = 1$, and the set of solutions is the set of multiples of $\boldsymbol{v}_2$ (eigenvector corresponding to $\lambda = 0$),

$$\left\{ \boldsymbol{v} \in \mathbb{R}^3 \;\middle|\; \exists \boldsymbol{p} \in \mathbb{R}^1,\ \boldsymbol{v} = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} \boldsymbol{p} \right\}$$

□

(Final June 18/19) Exercise 2(b) $\mathbf{A}$ tiene dos autoespacios, el asociado al autovalor $\lambda = 0$, y generado por $\boldsymbol{v}_2$, y el asociado al autovalor $\lambda = 1$, y generado por $\boldsymbol{v}_1$ y $\boldsymbol{v}_3$ (que es de dimensión 2 por ser $\boldsymbol{v}_1$ y $\boldsymbol{v}_3$

linealmente independientes)There are two eigenspaces, one for $\lambda = 0$ and another one for $\lambda = 1$ (this one with geometric multiplicity 2, since $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ are linearly independent).

A matrix is symmetric if and only if eigenspaces corresponding to different eigenvalues are orthogonal. Hence, $\mathbf{A}$ is symmetric if $\boldsymbol{v}_2$ is orthogonal to both $\boldsymbol{v}_1$ and $\boldsymbol{v}_3$. Lets see:

$$\boldsymbol{v}_2\left[\boldsymbol{v}_1;\quad \boldsymbol{v}_3\right] = \begin{pmatrix}-1, & 2, & 1,\end{pmatrix}\begin{bmatrix}2 & 2\\ 1 & 1\\ 0 & 1\end{bmatrix} = \begin{pmatrix}0, & 1,\end{pmatrix}$$

so $\boxed{\mathbf{A} \text{ is NOT symmetric}}$.

To prove that $\mathbf{A}$ is diagonalizable we need to check that $\boldsymbol{v}_1$, $\boldsymbol{v}_2$ and $\boldsymbol{v}_3$ are linearly independent, and therefore they form a basis for $\mathbb{R}^3$.

$$\left[\boldsymbol{v}_1;\quad 2\boldsymbol{v}_2;\quad \boldsymbol{v}_3\right] = \begin{bmatrix}2 & -2 & 2\\ 1 & 4 & 1\\ 0 & 2 & 1\end{bmatrix} \xrightarrow[{[(-1)\mathbf{1}+\mathbf{3}]}]{\substack{\boldsymbol{\tau}\\ {[(1)\mathbf{1}+\mathbf{2}]}}} \begin{bmatrix}2 & 0 & 0\\ 1 & 5 & 0\\ 0 & 2 & 1\end{bmatrix}$$

□

(Final June 18/19) Exercise 2(c) Aunque podríamos aplicar la proyección ortogonal de un vector sobre el espacio generado por el otro y tomar el vector diferencia; tenemos otra forma más sencilla de hacerlo. El segundo autovector $\boldsymbol{w}$ asociado a $\lambda = 1$ tiene que ser combinación lineal de $\boldsymbol{v}_1$ y $\boldsymbol{v}_3$, es decir $\boldsymbol{w} = a\boldsymbol{v}_1 + b\boldsymbol{v}_3$, por tantoAlthough we could apply the orthogonal projection of one vector onto the spam of the other, and then we can take take the difference vector; we are going to proceed in other way. We need a vector $\boldsymbol{w}$ in the eigenspace corresponding to $\lambda = 1$ (a linear combination of $\boldsymbol{v}_1$ y $\boldsymbol{v}_3$)

$$\boldsymbol{w} = \left[\boldsymbol{v}_1;\quad \boldsymbol{v}_3\right]\begin{pmatrix}a\\ b\end{pmatrix} = \begin{bmatrix}2 & 2\\ 1 & 1\\ 0 & 1\end{bmatrix}\begin{pmatrix}a\\ b\end{pmatrix};$$

and we want $\boldsymbol{w}$ to be perpendicular to one of them, for example perpendicular to $\boldsymbol{v}_1$; hence

$$\boldsymbol{v}_1 \cdot \boldsymbol{w} = 0; \quad \Rightarrow \quad \begin{pmatrix}2, & 1, & 0,\end{pmatrix}\begin{bmatrix}2 & 2\\ 1 & 1\\ 0 & 1\end{bmatrix}\begin{pmatrix}a\\ b\end{pmatrix} = 0; \quad \Rightarrow \quad 5\,(a+b) = 0; \quad \Rightarrow \quad b = -a.$$

So, for all $a \neq 0$, the vector $(a\boldsymbol{v}_1 - a\boldsymbol{v}_3)$ is an eigenvector corresponding to $\lambda = 1$ and orthogonal to $\boldsymbol{v}_1$. If, for example, $a = 1$, then we get $\boldsymbol{v}_1 - \boldsymbol{v}_3 = \begin{pmatrix}0, & 0, & -1,\end{pmatrix}$. Now, we only need to normalize in order to get an orthonormal basis: $\left[\begin{pmatrix}\frac{2\sqrt{5}}{5}\\ \frac{\sqrt{5}}{5}\\ 0\end{pmatrix};\quad \begin{pmatrix}0\\ 0\\ -1\end{pmatrix};\right]$.

□

(Final June 18/19) Exercise 2(d) Since $\boldsymbol{v}_1$ and $\boldsymbol{v}_3$ belong to the eigenspace corresponding to $\lambda = 1$, any linear combination of both belongs that eigenspace. Therefore $\mathbf{A}(2\boldsymbol{v}_1 - \boldsymbol{v}_3) = 1(2\boldsymbol{v}_1 - \boldsymbol{v}_3)$ and

$$\mathbf{A}^k(2\boldsymbol{v}_1 - \boldsymbol{v}_3) = 1^k(2\boldsymbol{v}_1 - \boldsymbol{v}_3) = (2\boldsymbol{v}_1 - \boldsymbol{v}_3) = \begin{pmatrix}2\\ 1\\ -1\end{pmatrix}.$$

□

(Final June 18/19) Exercise 2(e) $(2\boldsymbol{v}_1 - \boldsymbol{v}_3)\mathbf{A}(2\boldsymbol{v}_1 - \boldsymbol{v}_3) = \begin{pmatrix}2, & 1, & -1,\end{pmatrix}\begin{pmatrix}2\\ 1\\ -1\end{pmatrix} = 6.$

□

(Final June 18/19) Exercise 3(a) False:

- $\mathbf{P}$ is the matrix $\mathbf{P} = \mathbf{X}\left(\mathbf{X}^\intercal\mathbf{X}\right)^{-1}\mathbf{X}^\intercal$; so it is of order m, but it is the product of matrices $\mathbf{X}$, $\left(\mathbf{X}^\intercal\mathbf{X}\right)^{-1}$ and $\mathbf{X}^\intercal$, and all these matrices have rank $n < m$. It follows that rank of $\mathbf{P}$ can't be m.
- $\mathbf{X}$ has rank n, so $dim\,\mathcal{C}\left(\mathbf{X}\right) = n < m$. Hence, there are non-null vectors $\boldsymbol{y}$ in $\mathbb{R}^m$ orthogonal to $\mathcal{C}\left(\mathbf{X}\right)$, whose projection is $\mathbf{0}$. That is, $\mathbf{P}\boldsymbol{y} = \mathbf{0}$ for some $\boldsymbol{y} \neq \mathbf{0}$. So $\mathbf{P}$ *is singular*.

$\square$

(Final June 18/19) Exercise 3(b) **True**:

$$\mathbf{P}^\intercal = \left(\mathbf{X}\left(\mathbf{X}^\intercal\mathbf{X}\right)^{-1}\mathbf{X}^\intercal\right)^\intercal = \left(\mathbf{X}^\intercal\right)^\intercal\left(\left(\mathbf{X}^\intercal\mathbf{X}\right)^{-1}\right)^\intercal\mathbf{X}^\intercal = \mathbf{X}\left(\mathbf{X}^\intercal\mathbf{X}\right)^{-1}\mathbf{X}^\intercal = \mathbf{P};$$

where $\mathbf{X}^\intercal\mathbf{X}$ is symmetric, so its inverse is also symmetric $\Rightarrow$ $\left(\left(\mathbf{X}^\intercal\mathbf{X}\right)^{-1}\right)^\intercal = \left(\mathbf{X}^\intercal\mathbf{X}\right)^{-1}$.

$\square$

(Final June 18/19) Exercise 3(c) **True**:

$$\mathbf{PP} = \mathbf{X}\left(\mathbf{X}^\intercal\mathbf{X}\right)^{-1}\mathbf{X}^\intercal\cdot\mathbf{X}\left(\mathbf{X}^\intercal\mathbf{X}\right)^{-1}\mathbf{X}^\intercal = \mathbf{X}\left(\mathbf{X}^\intercal\mathbf{X}\right)^{-1}\left(\mathbf{X}^\intercal\mathbf{X}\right)\left(\mathbf{X}^\intercal\mathbf{X}\right)^{-1}\mathbf{X}^\intercal = \mathbf{X}\left(\mathbf{X}^\intercal\mathbf{X}\right)^{-1}\mathbf{X}^\intercal = \mathbf{P}$$

$\square$

(Final June 18/19) Exercise 3(d) **True**: We only need to check that $\mathbf{P}\boldsymbol{v}\cdot(\boldsymbol{v}-\mathbf{P}\boldsymbol{v})$ is zero:

$$\mathbf{P}\boldsymbol{v}\cdot(\boldsymbol{v}-\mathbf{P}\boldsymbol{v}) \quad = \quad \boldsymbol{v}\mathbf{P}^\intercal\cdot(\boldsymbol{v}-\mathbf{P}\boldsymbol{v}) \quad = \quad \boldsymbol{v}\mathbf{P}^\intercal\boldsymbol{v}-\boldsymbol{v}\mathbf{P}^\intercal\mathbf{P}\boldsymbol{v} \quad = \quad \boldsymbol{v}\mathbf{P}\boldsymbol{v}-\boldsymbol{v}\mathbf{P}\boldsymbol{v} \quad = \quad 0.$$

where $\mathbf{P}\boldsymbol{v} = \boldsymbol{v}\mathbf{P}^\intercal$, where $\mathbf{P}^\intercal = \mathbf{P}$ and therefore $\mathbf{P}^\intercal\mathbf{P} = \mathbf{PP} = \mathbf{P}$.

$\square$

(Final June 18/19) Short questions set 1(a)

$$\begin{aligned}
\mathbf{C}^2 =& \mathbf{CC} = \mathbf{CC}^\intercal && \text{since } \mathbf{C} = \mathbf{C}^\intercal\\
=& \mathbf{AB}(\mathbf{AB})^\intercal = \mathbf{ABB}^\intercal\mathbf{A}^\intercal && \text{since } (\mathbf{AB})^\intercal = \mathbf{B}^\intercal\mathbf{A}^\intercal\\
=& \mathbf{AIA}^\intercal && \text{because } \mathbf{BB}^\intercal = \mathbf{I} \text{ since } \mathbf{B} \text{ is orthogonal}\\
=& \mathbf{AA}^\intercal = \mathbf{I} && \text{since } \mathbf{A} \text{ is also orthogonal.}
\end{aligned}$$

$\square$

(Final June 18/19) Short questions set 1(b) **YES.** The coeficient matrix $\mathbf{A}$ is 2 by 4, and it has rank 2. Hence $\dim\mathcal{N}(\mathbf{A}) = 2$. And both vectors satify the system since

$$\begin{bmatrix}2 & 1 & -1 & 0\\ 1 & & & -1\end{bmatrix}\begin{pmatrix}1\\-2\\0\\1\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}; \qquad \begin{bmatrix}2 & 1 & -1 & 0\\ 1 & & & -1\end{bmatrix}\begin{pmatrix}1\\0\\2\\1\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix};$$

so both vectors belong to $\mathcal{N}(\mathbf{A})$; and both are linearly independent (note the location of the null components in both vectors). Since both are linearly independent vectors in a subspace of dimension 2, they are a basis for that subspace.

$\square$

(Final June 18/19) Short questions set 1(c) The spam is following the set:

$$\left\{\boldsymbol{x}\in\mathbb{R}^3 \;\middle|\; \exists a,b,c\in\mathbb{R},\; \begin{pmatrix}x\\y\\z\end{pmatrix} = a\begin{pmatrix}1\\2\\0\end{pmatrix} + b\begin{pmatrix}2\\1\\1\end{pmatrix} + c\begin{pmatrix}1\\-1\\1\end{pmatrix}\right\}.$$

By gaussian elimination we can find the required equations:

$$\left[\begin{array}{ccc}1 & 2 & 0\\ 2 & 1 & 1\\ 1 & -1 & 1\\ \hline x & y & z\end{array}\right] \xrightarrow[{[(-2)\mathbf{1}+\mathbf{2}]}]{\tau} \left[\begin{array}{ccc}1 & 0 & 0\\ 2 & -3 & 1\\ 1 & -3 & 1\\ \hline x & -2x+y & z\end{array}\right] \xrightarrow[{\substack{[(3)\mathbf{3}]\\ [(1)\mathbf{2}+\mathbf{3}]}}]{\tau} \left[\begin{array}{ccc}1 & 0 & 0\\ 2 & -3 & 0\\ 1 & -3 & 0\\ \hline x & -2x+y & -2x+y+3z\end{array}\right]$$

hence $\Rightarrow$ $\boxed{-2x+y+3z=0}$.

$\square$

(Final June 18/19) Short questions set 1(d)

$$\left\{\begin{pmatrix}x\\y\\z\end{pmatrix} \;\middle|\; \exists a\in\mathbb{R}: \begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}1\\-3\\1\end{pmatrix} + a\left(\begin{pmatrix}1\\-3\\1\end{pmatrix} - \begin{pmatrix}-2\\4\\5\end{pmatrix}\right)\right\} = \left\{\begin{pmatrix}x\\y\\z\end{pmatrix} \;\middle|\; \exists a\in\mathbb{R}: \begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}1\\-3\\1\end{pmatrix} + a\begin{pmatrix}3\\-7\\-4\end{pmatrix}\right\}$$

$\square$

(Final June 18/19) Short questions set 2(a) Since $\det \mathbf{A} = 2\det\begin{bmatrix}1&0&1\\1&-1&0\\0&-1&1\end{bmatrix} = -4 \neq 0$, the matrix is invertible. Using cofactors is easy to find the component $(3,2)$ of $\mathbf{A}^{-1}$:

$$\frac{1}{\det\mathbf{A}}\cdot C_{23} = \frac{-1}{4}\cdot(-1)^5\cdot\begin{vmatrix}0&1&1\\0&0&1\\-2&0&0\end{vmatrix} = \frac{-2}{4} = \boxed{\frac{-1}{2}}.$$

$\square$

(Final June 18/19) Short questions set 2(b) We can find that cordinate using the Cramer's rule:

$$x_3 = \frac{1}{\det\mathbf{A}}\begin{vmatrix}0&1&1&1\\0&1&3&0\\0&0&2&1\\-2&0&-1&0\end{vmatrix} = \frac{-1}{4}\cdot 2\cdot\begin{vmatrix}1&1&1\\1&3&0\\0&2&1\end{vmatrix} = \frac{-1}{4}\cdot 2\cdot 4 = \boxed{-2}.$$

$\square$

(Final June 18/19) Short questions set 3(a) Since $\mathbf{A} = \begin{bmatrix}a&0&1\\0&a&0\\1&0&a\end{bmatrix}$; is simétrica, it is always diagonalizable.

$\square$

(Final June 18/19) Short questions set 3(b) Since

$$\begin{bmatrix}a&0&1\\0&a&0\\1&0&a\end{bmatrix}\begin{pmatrix}1\\0\\1\end{pmatrix} = (a+1)\begin{pmatrix}1\\0\\1\end{pmatrix};\quad\text{and}\quad\begin{bmatrix}a&0&1\\0&a&0\\1&0&a\end{bmatrix}\begin{pmatrix}0\\1\\0\end{pmatrix} = a\begin{pmatrix}0\\1\\0\end{pmatrix},$$

both are eigenvectors of $\mathbf{A}$, the first corresponding to the eigenvalue $(a+1)$ and the second corresponding to a.

$\square$

(Final June 18/19) Short questions set 3(c) Since $(a+1)$ and a are two out of the three eigenvalues, and since the trace is $3a$, the third eigenvalue must be $(a-1)$, so: $(a+1)+a+(a-1) = 3a$. Hence, the three eigenvalues for $\mathbf{A}$ are: $(a-1)$, a and $(a+1)$:

$$\begin{cases} a>1 & q(\boldsymbol{x})>0 & \text{positive definite}\\ a=1 & q(\boldsymbol{x})\geq 0 & \text{positive semi-definite}\\ -1<a<1 & q(\boldsymbol{x})\lesseqgtr 0 & \text{nothing (indefinite)}\\ a=-1 & q(\boldsymbol{x})\leq 0 & \text{negative semi-definite}\\ a<-1 & q(\boldsymbol{x})<0 & \text{negative definite}\end{cases}$$

where $\lesseqgtr$ means "less, equal or greater than".

$\square$

(Final June 18/19) Short questions set 3(d) We can diagonalize by congruence: since

$$\mathbf{A} = \begin{bmatrix}2&0&1\\0&2&0\\1&0&2\end{bmatrix}\xrightarrow{\overset{\tau}{[(-\frac{1}{2})\mathbf{1}+\mathbf{3}]}}\begin{bmatrix}2&0&0\\0&2&0\\1&0&2-\frac{1}{2}\end{bmatrix}\xrightarrow[\overset{\tau}{[(-\frac{1}{2})\mathbf{1}+\mathbf{3}]}]{}\begin{bmatrix}2&0&0\\0&2&0\\0&0&2-\frac{1}{2}\end{bmatrix} = \mathbf{D};$$

then $\mathbf{A} = {}_{\overset{\tau}{[(-1/2)\mathbf{1}+\mathbf{3}]}}\mathbf{I}\cdot\mathbf{D}\cdot\mathbf{I}_{\overset{\tau}{[(-1/2)\mathbf{1}+\mathbf{3}]}}$ and therefore

$$q(\boldsymbol{x}) = \boldsymbol{x}\mathbf{A}\boldsymbol{x} = \begin{pmatrix}x&y&z\end{pmatrix}\begin{bmatrix}1&0&0\\0&1&0\\1/2&0&1\end{bmatrix}\begin{bmatrix}2&&\\&2&\\&&\frac{3}{2}\end{bmatrix}\begin{bmatrix}1&0&1/2\\0&1&0\\0&0&1\end{bmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix} = \boxed{2\left(x-\tfrac{z}{2}\right)^2+2\left(y\right)^2+\tfrac{3}{2}\left(z\right)^2}.$$

Or we can find an orthogonal diagonalization of $\mathbf{A}$. To do so, we need a third eigenvector that, since $\mathbf{A}$ is symmetric, it must be orthogonal to the both eigenvectors in part b).

$$\left[\begin{array}{ccc}1&0&1\\0&1&0\\\hline 1&0&0\\0&1&0\\0&0&1\end{array}\right]\xrightarrow[{[(-1)\mathbf{1}+\mathbf{3}]}]{\boldsymbol{\tau}}\left[\begin{array}{ccc}1&0&0\\0&1&0\\\hline 1&0&-1\\0&1&0\\0&0&1\end{array}\right]$$

And now we can use an orthogonal matrix $\mathbf{Q}$ whose columns are eigenvectors of $\mathbf{A}$:

$$\mathbf{Q}=\frac{1}{\sqrt{2}}\begin{bmatrix}1&0&-1\\0&\sqrt{2}&0\\1&0&1\end{bmatrix}$$

We known that $\mathbf{A}=\mathbf{QDQ}^{\intercal}$, where $\mathbf{D}$ is diagonal with eigenvalues 3, 2, 1 in the main diagonal (see part c); therefore

$$q(\boldsymbol{x})=\boldsymbol{x}\mathbf{A}\boldsymbol{x}=\frac{1}{\sqrt{2}}\begin{pmatrix}x&y&z\end{pmatrix}\begin{bmatrix}1&0&-1\\0&\sqrt{2}&0\\1&0&1\end{bmatrix}\begin{bmatrix}3&&\\&2&\\&&1\end{bmatrix}\begin{bmatrix}1&0&1\\0&\sqrt{2}&0\\-1&0&1\end{bmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}\frac{1}{\sqrt{2}}$$
$$=\boxed{\tfrac{3}{2}\big(x+z\big)^2+2\big(y\big)^2+\tfrac{1}{2}\big(-x+z\big)^2}.$$

□

(Final May 18/19) Exercise 1(a) Since $\boldsymbol{u}$ and $\boldsymbol{v}$ are no perpendicular, we need a linear combination $\big(a\boldsymbol{u}+b\boldsymbol{v}\big)$, orthogonal to one of them, for example orthogonal to $\boldsymbol{u}$. Hence:

$$\boldsymbol{u}\cdot\big(a\boldsymbol{u}+b\boldsymbol{v}\big)=0\quad\Rightarrow\quad\begin{pmatrix}1,&1,&0\end{pmatrix}\begin{bmatrix}1&0\\1&1\\0&1\end{bmatrix}\begin{pmatrix}a\\b\end{pmatrix}=0\quad\Rightarrow\quad\begin{pmatrix}2,&1\end{pmatrix}\begin{pmatrix}a\\b\end{pmatrix}=0\quad\Rightarrow\quad b=-2a.$$

For example, if $a=1$, then vector $\big(\boldsymbol{u}-2\boldsymbol{v}\big)=\begin{pmatrix}1\\1\\0\end{pmatrix}-2\begin{pmatrix}0\\1\\0\end{pmatrix}=\begin{pmatrix}1\\-1\\-2\end{pmatrix}$ is orthogonal to $\boldsymbol{u}$. Finally, we need to normalize both vectors $\rightarrow$ $\boxed{\text{Orthonormal basis for } \mathcal{S}\text{: } \left\{\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\\0\end{pmatrix},\ \frac{1}{\sqrt{6}}\begin{pmatrix}1\\-1\\-2\end{pmatrix}\right\}}.$

□

(Final May 18/19) Exercise 1(b) We need to solve $\mathbf{S}^{\intercal}\boldsymbol{x}=\mathbf{0}$, (where $\mathbf{S}=\big[\boldsymbol{uv}\big]$):

$$\left[\begin{array}{ccc}1&1&0\\0&1&1\\\hline 1&0&0\\0&1&0\\0&0&1\end{array}\right]\xrightarrow[{[(-1)\mathbf{2}+\mathbf{3}]}]{\underset{[(-1)\mathbf{1}+\mathbf{2}]}{\boldsymbol{\tau}}\ \boldsymbol{\tau}}\left[\begin{array}{ccc}1&0&0\\0&1&0\\\hline 1&-1&1\\0&1&-1\\0&0&1\end{array}\right]\quad\Rightarrow\quad\boxed{\begin{pmatrix}x\\y\\z\end{pmatrix}=a\begin{pmatrix}1\\-1\\1\end{pmatrix};\ \forall a\in\mathbb{R}.}$$

□

(Final May 18/19) Exercise 1(c) If $\mathbf{S}$ is the matrix $\big[\boldsymbol{u}\,\boldsymbol{v}\big]$, then $\mathbf{P}=\mathbf{S}\big(\mathbf{S}^{\intercal}\mathbf{S}\big)^{-1}\mathbf{S}^{\intercal}$. Since the non-null vectors in $\mathcal{S}$ are $\boldsymbol{y}=\mathbf{S}\boldsymbol{x}$ (with $\boldsymbol{x}\neq\mathbf{0}$), then:

$$\boxed{\mathbf{P}\boldsymbol{y}=\mathbf{S}\big(\mathbf{S}^{\intercal}\mathbf{S}\big)^{-1}\mathbf{S}^{\intercal}\boldsymbol{y}=\mathbf{S}\big(\mathbf{S}^{\intercal}\mathbf{S}\big)^{-1}\mathbf{S}^{\intercal}\mathbf{S}\boldsymbol{x}=\mathbf{S}\boldsymbol{x}=1\boldsymbol{y}}.$$

And since the non-null vectors $\boldsymbol{z}$ in $\mathcal{S}^{\perp}$ satify $\mathbf{S}^{\intercal}\boldsymbol{z}=\mathbf{0}$, then:

$$\boxed{\mathbf{P}\boldsymbol{z}=\mathbf{S}\big(\mathbf{S}^{\intercal}\mathbf{S}\big)^{-1}\mathbf{S}^{\intercal}\boldsymbol{z}=\mathbf{S}\big(\mathbf{S}^{\intercal}\mathbf{S}\big)^{-1}\mathbf{0}=\mathbf{0}=0\boldsymbol{z}}.$$

Resolución alternativa, si tomamos una matriz $\mathbf{X}$ cuyas columnas son una base ortonormal de $\mathcal{S}$ [como la del apartado a)]; entonces la matriz proyección es

$$\mathbf{P}=\mathbf{XX}^{\intercal}=\frac{1}{3}\begin{bmatrix}2&1&-1\\1&2&1\\-1&1&2\end{bmatrix}.$$

Con esta matriz es inmediato comprobar que $\mathbf{P}\boldsymbol{w} = \mathbf{0}$, donde $\boldsymbol{w} \in \mathcal{S}^\perp$ [por ejemplo la solución especial encontrada en el apartado b)].

Y también que $\mathbf{P}\boldsymbol{u} = \boldsymbol{u}$, $\mathbf{P}\boldsymbol{v} = \boldsymbol{v}$; Así, si $\boldsymbol{x} \in \mathcal{S}$ existen $a, b \in \mathbb{R}$ tales que $\boldsymbol{x} = a\boldsymbol{u} + b\boldsymbol{v}$ y entonces $\mathbf{P}\boldsymbol{x} = \mathbf{P}\big(a\boldsymbol{u} + b\boldsymbol{v}\big) = a\mathbf{P}\boldsymbol{u} + b\mathbf{P}\boldsymbol{v} = a\boldsymbol{u} + b\boldsymbol{v} = \boldsymbol{x}$.

□

(Final May 18/19) Exercise 1(d) The eigenspace corresponding to $\lambda = 1$ consists of all non-null vectors in $\mathcal{S}$, and the eigenspace corresponding to $\lambda = 0$ consists of all non-null vectors in $\mathcal{S}^\perp$. We alredy known an orthonormal basis for $\mathcal{S}$; and we already known that $(1, -1, 1)$ is a basis for $\mathcal{S}^\perp$. Hence

$$\boxed{\mathbf{Q} = \left[\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\\0\end{pmatrix}\quad \frac{1}{\sqrt{6}}\begin{pmatrix}1\\-1\\-2\end{pmatrix}\quad \frac{1}{\sqrt{3}}\begin{pmatrix}1\\-1\\1\end{pmatrix}\right] = \begin{bmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}}\\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{6}} & \frac{-1}{\sqrt{3}}\\ 0 & \frac{-2}{\sqrt{6}} & \frac{1}{\sqrt{3}}\end{bmatrix}; \qquad \mathbf{D} = \begin{bmatrix}1 & & \\ & 1 & \\ & & 0\end{bmatrix}}.$$

□

(Final May 18/19) Exercise 2(a)

$$\left[\begin{array}{cccc|c}1 & -1 & 1 & 2 & -b\\ 1 & 0 & 1 & 2 & -0\\ a & 1 & 1 & 2 & -0\\ \hline 1 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1 & 0\end{array}\right] \xrightarrow[\substack{(1)\mathbf{1}+\mathbf{2}\\(-1)\mathbf{1}+\mathbf{3}\\(b)\mathbf{1}+\mathbf{5}}]{\substack{\tau\\[(-2)\mathbf{3}+\mathbf{4}]}} \left[\begin{array}{cccc|c}1 & 0 & 0 & 0 & 0\\ 1 & 1 & 0 & 0 & b\\ a & 1+a & 1-a & 0 & ab\\ \hline 1 & 1 & -1 & 0 & b\\ 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 1 & -2 & 0\\ 0 & 0 & 0 & 1 & 0\end{array}\right] \xrightarrow[{[(-b)\mathbf{2}+\mathbf{5}]}]{\tau} \left[\begin{array}{cccc|c}1 & 0 & 0 & 0 & 0\\ 1 & 1 & 0 & 0 & 0\\ a & 1+a & 1-a & 0 & -b\\ \hline 1 & 1 & -1 & 0 & 0\\ 0 & 1 & 0 & 0 & -b\\ 0 & 0 & 1 & -2 & 0\\ 0 & 0 & 0 & 1 & 0\end{array}\right]$$

Discussion: $\boxed{\begin{cases} b = 0 & \text{solvable (with infinite solutions)}\\ b \neq 0 & \begin{cases} a \neq 1 & \text{solvable (with infinite solutions)}\\ a = 1 & \textbf{un}\text{solvable}\end{cases}\end{cases}}$

□

(Final May 18/19) Exercise 2(b) $\boxed{\left\{\boldsymbol{x} \in \mathbb{R}^4 \;\middle|\; \begin{pmatrix}x_1\\x_2\\x_3\\x_4\end{pmatrix} = \alpha\begin{pmatrix}-1\\0\\1\\0\end{pmatrix} + \beta\begin{pmatrix}0\\0\\-2\\1\end{pmatrix}, \quad \forall \alpha, \beta \in \mathbb{R}\right\}}.$

□

(Final May 18/19) Exercise 2(c) The set $\{(-1, 0, 1, 0),\ (0, 0, -2, 1)\}$ is a basis for the set of solutions since: both vectors are linearly independent and both are solutions (two linearly independent vectors in a subspace of dimension 2)

On the other hand

$$\left[\begin{array}{cc|c}-1 & 0 & -1\\ 0 & 0 & -0\\ 1 & -2 & -1\\ 0 & 1 & 1\\ \hline 1 & 0 & 0\\ 0 & 1 & 0\end{array}\right] \xrightarrow[{[(-1)\mathbf{1}+\mathbf{3}]}]{\tau} \left[\begin{array}{cc|c}-1 & 0 & 0\\ 0 & 0 & 0\\ 1 & -2 & -2\\ 0 & 1 & 1\\ \hline 1 & 0 & -1\\ 0 & 1 & 0\end{array}\right] \xrightarrow[{[(-1)\mathbf{2}+\mathbf{3}]}]{\tau} \left[\begin{array}{cc|c}-1 & 0 & 0\\ 0 & 0 & 0\\ 1 & -2 & 0\\ 0 & 1 & 0\\ \hline 1 & 0 & -1\\ 0 & 1 & -1\end{array}\right]$$

So $\boxed{\text{the coordinates are } (-1, -1)}$, that is, $\quad (-1)\begin{pmatrix}-1\\0\\1\\0\end{pmatrix} + (-1)\begin{pmatrix}0\\0\\-2\\1\end{pmatrix} = \begin{pmatrix}1\\0\\1\\-1\end{pmatrix}.$

□

(Final May 18/19) Exercise 2(d)

Yes. Applying a different sequence of transformations of the columns of the coefficient matrix we get a different description of the same solution set:

$$\left[\begin{array}{cccc}1 & -1 & 1 & 2\\ 1 & 0 & 1 & 2\\ 1 & 1 & 1 & 2\\ \hline 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{array}\right] \xrightarrow[\substack{\tau\\[(-2)\mathbf{3}+\mathbf{4}]}]{\substack{\tau\\[(-1)\mathbf{3}+\mathbf{1}]}} \left[\begin{array}{cccc}0 & -1 & 1 & 0\\ 0 & 0 & 1 & 0\\ 0 & 1 & 1 & 0\\ \hline 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ -1 & 0 & 1 & -2\\ 0 & 0 & 0 & 1\end{array}\right] \Rightarrow \left\{\boldsymbol{x} \in \mathbb{R}^4 \;\middle|\; \begin{pmatrix}x_1\\x_2\\x_3\\x_4\end{pmatrix} = \alpha\begin{pmatrix}1\\0\\-1\\0\end{pmatrix} + \beta\begin{pmatrix}0\\0\\-2\\1\end{pmatrix}, \quad \forall \alpha, \beta \in \mathbb{R}\right\}$$

hence, $\alpha = x_1$ and $\beta = x_4$ and therefore $\begin{cases} x_2 & = 0 \\ & x_3 = -x_1 - 2x_4 \end{cases}$.

Una discusión un poco más larga. Nótese que si formamos una matriz cuyas columnas son los vectores de la base empleada para describir del conjunto de soluciones en el apartado (b), y la ampliamos con una solución particular, podemos expresar las soluciones en función de aquellas variables para las que (tras una serie de transformaciones elementales de las columnas) el correspondiente coeficiente alguna de las columnas es 1 y cero en el resto. Así, para comprobar que x_3 y x_4 pueden ser simultáneamente exógenas (algo que ya sabemos) realizamos la siguiente transformación:

$$\left[\begin{array}{cc|c} -1 & 0 & 0 \\ 0 & 0 & -b \\ 1 & -2 & 0 \\ 0 & 1 & 0 \end{array}\right] \xrightarrow[{[(2)\mathbf{1}+\mathbf{2}]}]{\boldsymbol{\tau}} \left[\begin{array}{cc|c} -1 & -2 & 0 \\ 0 & 0 & -b \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right]$$

De donde se deduce (recordando que $b = 0$) que las soluciones se pueden expresar como

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \alpha \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} -2 \\ 0 \\ 0 \\ 1 \end{pmatrix} \qquad \Rightarrow (\alpha = x_3,\ \beta = x_4) \begin{cases} x_1 & = -x_3 - 2x_4 \\ & x_2 = 0 \end{cases}$$

Es decir, hemos despejado x_1 y x_2 (variables endógenas) en función de x_3 y x_4 (variables exógenas).

Pero en el enunciado nos piden despejar x_2 y x_3 (variables endógenas) en función de x_1 y x_4 (variables exógenas). Esto será posible si mediante transformaciones elementales podemos convertir el coeficiente correspondiente a cada una de ellas en un 1 en alguna de las columnas y en cero en el resto,

$$\left[\begin{array}{cc|c} -1 & 0 & 0 \\ 0 & 0 & -b \\ 1 & -2 & 0 \\ 0 & 1 & 0 \end{array}\right] \xrightarrow[{[(-1)\mathbf{1}]}]{\boldsymbol{\tau}} \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 0 & -b \\ -1 & -2 & 0 \\ 0 & 1 & 0 \end{array}\right]$$

De donde se deduce (recordando que $b = 0$) que las soluciones se pueden expresar como

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \alpha \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 0 \\ 0 \\ -2 \\ 1 \end{pmatrix} \qquad \Rightarrow (\alpha = x_1,\ \beta = x_4) \begin{cases} x_2 & = 0 \\ & x_3 = -x_1 - 2x_4 \end{cases}$$

Es decir, hemos despejado x_2 y x_3 (variables endógenas) en función de x_1 y x_4 (variables exógenas).

$\boxed{\text{Por tanto la respuesta es que } x_1 \text{ y } x_4 \textbf{ si} \text{ pueden ser simultáneamente exógenas}}$.

Otra alternativa: El rango de $\mathbf{A}$ es dos, y también es dos el rango de la submatriz formada la segunda y tercera columnas de $\mathbf{A}$, luego para cualquier valor de las variables $x_!$ y x_4 se cumple que es

□

(Final May 18/19) Exercise 3(a) The solution set is a subspace if the system is homogeneous. Since (S1) is equivalent to $\mathbf{A}\boldsymbol{x} = \boldsymbol{b} + \alpha \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$, $\boxed{\text{the solution set is a subspace if } \boldsymbol{b} = -\alpha \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix},\ \forall \alpha \in \mathbb{R}.}$

□

(Final May 18/19) Exercise 3(b) $\begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix} \xrightarrow[\substack{[(-1)\mathbf{4}+\mathbf{3}] \\ (1)\mathbf{3}+\mathbf{2} \\ (-1)\mathbf{2}+\mathbf{1}}]{\boldsymbol{\tau}} \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \implies \boxed{\operatorname{rg}(\mathbf{A}) = 3}$.

□

(Final May 18/19) Exercise 3(c) We are asked to verify if $\boldsymbol{v}$ and $\boldsymbol{w}$ are eigenvalues of $\mathbf{A}$. Lets see

$$\mathbf{A}\boldsymbol{v} = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 2 \\ 2 \end{pmatrix} = 2\boldsymbol{v}; \qquad \text{y} \qquad \mathbf{A}\boldsymbol{w} = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix} \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = 0\boldsymbol{w}.$$

Hence $\boxed{\boldsymbol{v} \text{ is a solution when } \alpha = 2 \text{ and } \boldsymbol{w} \text{ is a solution when } \alpha = 0}$.

□

(Final May 18/19) Exercise 3(d) We are asked to prove that $\mathbf{A}^3\boldsymbol{u}$ is a multiple of $\boldsymbol{v}$. Since $\boldsymbol{v}$ and $\boldsymbol{w}$ are eigenvectors corresponding to $\lambda = 2$ and $\lambda = 0$ respectively, we get

$$\boxed{\mathbf{A}^3\boldsymbol{u} = \mathbf{A}^3(p\boldsymbol{v} + q\boldsymbol{w}) = p\mathbf{A}^3\boldsymbol{v} + q\mathbf{A}^3\boldsymbol{w} = p\left(2^3 \cdot \boldsymbol{v}\right) + q\left(0^3 \cdot \boldsymbol{w}\right) = (8p)\boldsymbol{v}}.$$

□

(Final May 18/19) Exercise 3(e) $\begin{pmatrix} x & y & z & w \end{pmatrix} \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix} \begin{pmatrix} x & y & z & w \end{pmatrix} = \boxed{y^2 + z^2 + 2xz + 2xw + 2yw}$.

Diagonalizing by congruence:

$$\begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix} \xrightarrow{\overset{\tau}{[(1)\mathbf{4}+\mathbf{1}]}} \begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix} \xrightarrow[\overset{\tau}{[(1)\mathbf{4}+\mathbf{1}]}]{} \begin{bmatrix} 2 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix} \xrightarrow{\substack{\overset{\tau}{[(-\frac{1}{2})\mathbf{2}+\mathbf{1}]} \\ (-\frac{1}{2})\mathbf{3}+\mathbf{1} \\ (-\frac{1}{2})\mathbf{4}+\mathbf{1}}} \begin{bmatrix} 2 & 0 & 0 & 0 \\ 1 & 1/2 & -1/2 & 1/2 \\ 1 & -1/2 & 1/2 & -1/2 \\ 1 & 1/2 & -1/2 & -1/2 \end{bmatrix}$$

$$\xrightarrow[\substack{\overset{\tau}{[(-\frac{1}{2})\mathbf{4}+\mathbf{1}]} \\ (-\frac{1}{2})\mathbf{3}+\mathbf{1} \\ (-\frac{1}{2})\mathbf{2}+\mathbf{1}}]{} \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 1/2 & -1/2 & 1/2 \\ 0 & -1/2 & 1/2 & -1/2 \\ 0 & 1/2 & -1/2 & -1/2 \end{bmatrix} \xrightarrow{\substack{\overset{\tau}{[(+1)\mathbf{3}+\mathbf{2}]} \\ (-1)\mathbf{4}+\mathbf{2}}} \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 0 \\ 0 & -1/2 & 0 & 0 \\ 0 & 1/2 & 0 & -1 \end{bmatrix} \xrightarrow{\substack{\overset{\tau}{[(-1)\mathbf{4}+\mathbf{2}]} \\ (+1)\mathbf{3}+\mathbf{2}}} \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

So it is $\boxed{\text{Indefinite}}$

Alterntivamente: de la expresión polinómica se observa que si $y = z = 0$, la forma cuadrática se reduce a $q(x, 0, 0, w) = 2xw$, que evidentemente puede tomar valores tanto positivos como negativos.

□

(Final May 18/19) Short questions set 1(a)

$$\mathbf{P} = \begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \end{bmatrix} \left(\begin{bmatrix} 1 & 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \end{bmatrix} \right)^{-1} \begin{bmatrix} 1 & 1 & -1 & 1 \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 1 & 1 & -1 & 1 \\ 1 & 1 & -1 & 1 \\ -1 & -1 & 1 & -1 \\ 1 & 1 & -1 & 1 \end{bmatrix}.$$

□

(Final May 18/19) Short questions set 1(b) $\begin{pmatrix} x, & y, & x \end{pmatrix} = \begin{pmatrix} 0, & 0, & 1 \end{pmatrix} + \alpha \begin{pmatrix} 1, & 2, & 4 \end{pmatrix}$ for all $\alpha \in \mathbb{R}$.

□

(Final May 18/19) Short questions set 1(c) The product must be $\mathbf{E}_1\mathbf{A}\mathbf{E}_2 = \begin{bmatrix} 1 & & \\ -1 & 1 & \\ & & 1 \end{bmatrix} \mathbf{A} \begin{bmatrix} 1 & & \\ & 4 & \\ & & 1 \end{bmatrix}$.

Since matrix multiplication is **associative**, then: $\left(\mathbf{E}_1\mathbf{A}\right)\mathbf{E}_2 = \mathbf{E}_1\left(\mathbf{A}\mathbf{E}_2\right)$ (hence, we always get the same result).

(Mostrar un ejemplo no es suficiente, es necesario aludir a la propiedad asociativa del producto)

□

(Final May 18/19) Short questions set 1(d) EvidentementeSince $\left(\mathbf{A} - 2\mathbf{I}\right) = \begin{bmatrix} 1 & 0 & 1 & -1 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 2 & 0 & 2 & -2 \end{bmatrix}$ is singular, $\lambda = 2$ is an eigenvalue for A.

$$\left[\begin{array}{cccc}1&0&1&-1\\0&-1&1&0\\0&0&0&0\\2&0&2&-2\\\hline 1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{array}\right] \xrightarrow[(1)\mathbf{1}+\mathbf{4}]{\underset{[(-1)\mathbf{1}+\mathbf{3}]}{\tau}} \left[\begin{array}{cccc}1&0&0&0\\0&-1&1&0\\0&0&0&0\\2&0&0&0\\\hline 1&0&-1&1\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{array}\right] \xrightarrow{\underset{[(1)\mathbf{2}+\mathbf{3}]}{\tau}} \left[\begin{array}{cccc}1&0&0&0\\0&-1&0&0\\0&0&0&0\\2&0&0&0\\\hline 1&0&-1&1\\0&1&1&0\\0&0&1&0\\0&0&0&1\end{array}\right] \Longrightarrow \mathcal{L}\left(\left(\begin{pmatrix}-1\\1\\1\\0\end{pmatrix};\begin{pmatrix}1\\0\\0\\1\end{pmatrix}\right)\right) - \mathbf{0}$$

The corresponding eigenspace is set of non-null vectors in the span of $(-1, 1, 1, 0)$ and $(1, 0, 0, 1)$.

□

(Final May 18/19) Short questions set 2(a) **False.** Since $\left(\mathsf{B}^{-1}\right)^{\mathsf{T}} = \left(\mathsf{B}^{\mathsf{T}}\right)^{-1} = \left(\mathsf{B}\right)^{-1}$; the inverse of a symmetric matrix is also symmetric. But the product of two symmetric matrices is not symmetric in general. For example:

$$\begin{bmatrix}1&0\\0&2\end{bmatrix}\begin{bmatrix}0&1\\1&0\end{bmatrix} = \begin{bmatrix}0&1\\2&0\end{bmatrix}.$$

□

(Final May 18/19) Short questions set 2(b) **True.** An orthogonal matrix is square with orthonormal columns. If columns are orthogonal, they are linearly *independent*. But $\mathsf{A}\boldsymbol{x} = \mathbf{0}$ has infinite many solutions, and therefore the columns of A are linearly *dependent* (so they can not be orthogonal).

□

(Final May 18/19) Short questions set 2(c) **True.**

- $\left(\mathsf{I} - \mathsf{P}\right)^{\mathsf{T}} = \left(\mathsf{I}^{\mathsf{T}} - \mathsf{P}^{\mathsf{T}}\right) = \left(\mathsf{I} - \mathsf{P}\right)$.
- $\left(\mathsf{I} - \mathsf{P}\right)^2 = \left(\mathsf{I} - \mathsf{P}\right)\left(\mathsf{I} - \mathsf{P}\right) = \mathsf{I} - 2\mathsf{P} + \mathsf{P}^2 = \mathsf{I} - 2\mathsf{P} + \mathsf{P} = \left(\mathsf{I} - \mathsf{P}\right)$.

If P is the projection matrix onto a subspace $\mathcal{S}$ in $\mathbb{R}^n$, then $\left(\mathsf{I} - \mathsf{P}\right)$ is the projection matrix onto the orthogonal complement $\mathcal{S}^{\perp}$.

□

(Final May 18/19) Short questions set 2(d) **False.** Since $[\boldsymbol{v},\ \boldsymbol{w},\ \boldsymbol{u}]\begin{bmatrix}2&0&1\\0&1&1\\0&1&1\end{bmatrix} = [2\boldsymbol{v},\ (\boldsymbol{w}+\boldsymbol{u}),\ (\boldsymbol{v}+\boldsymbol{w}+\boldsymbol{u})]$ and the numeric matrix has rank 2, that system is dependent (It can't be a basis). We get the same conclusion by gaussian elimination:

$$[2\boldsymbol{v},\ (\boldsymbol{w}+\boldsymbol{u}),\ (\boldsymbol{v}+\boldsymbol{w}+\boldsymbol{u})] \xrightarrow{\underset{[(1/2)\mathbf{1}]}{\tau}} [\boldsymbol{v},\ (\boldsymbol{w}+\boldsymbol{u}),\ (\boldsymbol{v}+\boldsymbol{w}+\boldsymbol{u})] \xrightarrow[\underset{[(-1)\mathbf{2}+\mathbf{3}]}{\tau}]{\underset{[(-1)\mathbf{1}+\mathbf{3}]}{\tau}} [\boldsymbol{v},\ (\boldsymbol{w}+\boldsymbol{u}),\ \mathbf{0}]$$

□

(Final May 18/19) Short questions set 2(e) **True.** Since A is symmetric, it is orthogonally diagonalizable ($\mathsf{A} = \mathsf{Q}\mathsf{D}\mathsf{Q}^{\mathsf{T}}$). Since all numbers in the main diagonal of D are non-zero, D is invertible; hence A (a product of non-singular matrices) is invertible:

$$\mathsf{A}^{-1} = \left(\mathsf{Q}\mathsf{D}\mathsf{Q}^{\mathsf{T}}\right)^{-1} = \left(\mathsf{Q}^{\mathsf{T}}\right)^{-1}\mathsf{D}^{-1}\mathsf{Q}^{-1} = \mathsf{Q}\mathsf{D}^{-1}\mathsf{Q}^{\mathsf{T}} \qquad (\text{since } \mathsf{Q}^{\mathsf{T}} = \mathsf{Q}^{-1})$$

where D^{-1} is diagonal and the entries in the main diagonal are the reciprocal of the entries in the main diagonal of D, therefore all are positive. Hence, A^{-1} is symetric (since it is orthogonally diagonalizable) and positive definite.

□

(Final May 18/19) Short questions set 2(f) **False.** Row operations preserve the row space but, in general, row operations change the column space. For example

$$\begin{bmatrix}1&1&1\\1&1&1\end{bmatrix} \xrightarrow{\text{substracting the first row from the second one}} \begin{bmatrix}1&1&1\\0&0&0\end{bmatrix}$$

The column space in the first matrix is the span of $(1,1)$, but the column space of the second matrix is the span of $(1,0)$.

$\square$

(Final June 17/18) Exercise 1(a) We known the rank of a matrix do not change after some elementary transformations, hence

$$\mathsf{A} = \begin{bmatrix} \boldsymbol{v}_1, & \boldsymbol{v}_2, & \boldsymbol{v}_3, & (\boldsymbol{v}_1 + 2\boldsymbol{v}_2 + \boldsymbol{v}_4) \end{bmatrix} \xrightarrow[]{\substack{\boldsymbol{\tau} \\ [(-1)\mathbf{1}+\mathbf{4}] \\ (-2)\mathbf{2}+\mathbf{4}}} \begin{bmatrix} \boldsymbol{v}_1, & \boldsymbol{v}_2, & \boldsymbol{v}_3, & \boldsymbol{v}_4 \end{bmatrix},$$

indicates that $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3, \boldsymbol{v}_4\}$ is a linear independent set.

Hence, since $(\boldsymbol{v}_1 + 2\boldsymbol{v}_2 + \boldsymbol{v}_4)$ is a vector in $\mathbb{R}^4$, the set $\{\boldsymbol{v}_1,\ \boldsymbol{v}_2,\ \boldsymbol{v}_3,\ (\boldsymbol{v}_1 + 2\boldsymbol{v}_2 + \boldsymbol{v}_4)\}$ is a linear independent set of vectors in $\mathbb{R}^4$ (that is a subspace of dimension 4).

Therefore, $\boxed{\text{the set is a basis for } \mathbb{R}^4}$.

$\square$

(Final June 17/18) Exercise 1(b) Since both are linearly independent, $\boxed{\text{the spam has dimension 2}}$.

$\square$

(Final June 17/18) Exercise 1(c) We can answer using determinants: $\begin{vmatrix} 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \end{vmatrix} = -1 \begin{vmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{vmatrix} = -1(1) = -1 \neq 0$. hence, we have 4 vectors lineraly independent in $\mathbb{R}^4$ and therefore, since the dimension of $\mathbb{R}^4$ is 4, the set is a basis. Now, aplying Cramer's rule we get: $x_3 = \frac{1}{\det \mathsf{A}} \begin{vmatrix} 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{vmatrix} = \frac{1}{\det \mathsf{A}} \begin{vmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} = \frac{1}{\det \mathsf{A}} = -1.$

$\square$

(Final June 17/18) Exercise 1(d) The linear span is $\left\{ \boldsymbol{x} \;\middle|\; \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = a \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} + b \begin{pmatrix} 1 \\ 0 \\ -1 \\ 1 \end{pmatrix} ; \forall a, b \in \mathbb{R} \right\}$.

Since it is a two dimensional subspace of $\mathbb{R}^4$, we only need to multiply (using dot products) $\boldsymbol{x}$, $\boldsymbol{v}_2$ and $\boldsymbol{v}_3$ by two vectors in $\mathbb{R}^4$ that are orthogonal to $\boldsymbol{v}_2$ and $\boldsymbol{v}_3$. We already known that when applying gaussian elimination by columns, if we get zero columns is because we are multiplying the rows by perpendicular vectors ("the special solutions"). Hence, if we write $\boldsymbol{x}$, $\boldsymbol{v}_2$ and $\boldsymbol{v}_3$ as rows and the we apply gaussian elimination we get:

$$\left[\begin{array}{cccc} x & y & z & w \\ \hline 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 \\ 1 & 0 & -1 & 1 \end{array}\right] \xrightarrow[]{\substack{\boldsymbol{\tau} \\ [(1)\mathbf{1}+\mathbf{3}]}} \left[\begin{array}{cccc} x & {\color{red}y} & {\color{red}(z+x)} & w \\ \hline 0 & {\color{red}0} & {\color{red}0} & 0 \\ \hline 0 & {\color{blue}0} & {\color{blue}0} & 1 \\ 1 & {\color{blue}0} & {\color{blue}0} & 1 \end{array}\right] \Rightarrow \boxed{\begin{cases} \quad y & = 0 \\ x \quad + z & = 0 \end{cases}}.$$

$\square$

(Final June 17/18) Exercise 1(e) Now we need to find three linearly independent vectors in $\mathbb{R}^4$ that are orthogonal to $(1,1,0,0)$. Again, we can find an answer applying gaussian elimination by columns:

$$\left[\begin{array}{cccc} 1 & 1 & 0 & 0 \\ \hline 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow[]{\substack{\boldsymbol{\tau} \\ [(-1)\mathbf{1}+\mathbf{2}]}} \left[\begin{array}{cccc} 1 & {\color{blue}0} & {\color{blue}0} & {\color{blue}0} \\ \hline 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

Hence, the following is a parametric equation of this hyperplane

$$\left\{ \boldsymbol{x} \;\middle|\; \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 1 \end{pmatrix} + a \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix} + b \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} + c \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} ; \forall a, b, c \in \mathbb{R} \right\}.$$

Observación adicional: Nótese que $(1,-1,0,1)$ es perpendicular a $(1,1,0,0)$ y que por tanto el punto $(1,-1,0,1)$ es combinación lineal de los tres vectores perpendiculares a $(1,1,0,0)$. En particular, si tomamos $a=b=c=-1$ comprobamos que $\mathbf{0}$ pertenece a dicho hiperplano. Esto quiere decir que el conjunto es un *subespacio* de $\mathbb{R}^4$, y en consecuencia podemos escribir unas ecuaciones paramétricas más sencillas:

$$\left\{ \boldsymbol{x} \;\middle|\; \begin{pmatrix} x\\ y\\ z\\ w \end{pmatrix} = a\begin{pmatrix} 1\\ -1\\ 0\\ 0 \end{pmatrix} + b\begin{pmatrix} 0\\ 0\\ 1\\ 0 \end{pmatrix} + c\begin{pmatrix} 0\\ 0\\ 0\\ 1 \end{pmatrix} ;\forall a,b,c\in\mathbb{R} \right\}.$$

□

(Final June 17/18) Exercise 2(a)

$$\begin{bmatrix} 0&-1&0\\ -1&3&1\\ 0&1&2 \end{bmatrix} \xrightarrow{\underset{[1\rightleftharpoons 2]}{\tau}} \begin{bmatrix} -1&0&0\\ 3&-1&1\\ 1&0&2 \end{bmatrix} \xrightarrow{\underset{[(1)2+3]}{\tau}} \begin{bmatrix} -1&0&0\\ 3&-1&0\\ 1&0&2 \end{bmatrix}$$

so $\mathbf{A}\mathbf{I}_{\underset{[1\rightleftharpoons 2]}{\tau}}\mathbf{I}_{\underset{[(1)2+3]}{\tau}} = \begin{bmatrix} -1&0&0\\ 3&-1&0\\ 1&0&2 \end{bmatrix}$, where $\mathbf{I}_{\underset{[(1)2+3]}{\tau}} = \begin{bmatrix} 1&0&0\\ 0&1&1\\ 0&0&1 \end{bmatrix}$.

□

(Final June 17/18) Exercise 2(b) On the one hand, $\mathbf{A}$ is invertible if $|\mathbf{A}| = 6a-a-2 = 5a-2 \neq 0$. On the other hand we also need $\mathbf{A}\begin{pmatrix} -1\\ 0\\ 0 \end{pmatrix} = \begin{pmatrix} 0\\ 1\\ 0 \end{pmatrix}$; therefore $\boxed{a=0}$.

□

(Final June 17/18) Exercise 2(c) $|\mathbf{A}-3\mathbf{I}| = \begin{vmatrix} a-3&-1&0\\ -1&0&1\\ 0&1&-1 \end{vmatrix} = -(a-4) = 0,$ so $\boxed{a=4}$.

□

(Final June 17/18) Exercise 2(d) Since the additon of all eigenvalues equals the trace, $7 = 3+\lambda_3$. Therefore $\boxed{\lambda_3 = 4}$, and $\mathbf{A}-4\mathbf{I} = \begin{bmatrix} -2&-1&0\\ -1&-1&1\\ 0&1&-2 \end{bmatrix}$.

$$\left[\begin{array}{ccc} -2&-1&0\\ -1&-1&1\\ 0&1&-2\\ \hline 1&0&0\\ 0&1&0\\ 0&0&1 \end{array}\right] \xrightarrow{\underset{[(-2)2+1]}{\tau}} \left[\begin{array}{ccc} 0&-1&0\\ 1&-1&1\\ -2&1&-2\\ \hline 1&0&0\\ -2&1&0\\ 0&0&1 \end{array}\right] \xrightarrow{\underset{[(-1)1+3]}{\tau}} \left[\begin{array}{ccc} 0&-1&0\\ 1&-1&0\\ -2&1&0\\ \hline 1&0&-1\\ -2&1&2\\ 0&0&1 \end{array}\right], \text{ so}$$

$\boxed{\text{the eigenspace for } \lambda=4 \text{ is the linear span of } (-1,2,1)}$.

□

(Final June 17/18) Exercise 2(e) $\mathbf{A}$ is posite definite if and only if all principal minors $D_1 = a$, $D_2 = 3a-1$, $D_3 = 5a-2$ are positive, so $\boxed{a>\frac{2}{5}}$.

We find the same result analyzing the pivots of the echelon form if we do not multiply columns by negative numbers. $\mathbf{A}$ is posite definite if and only if all pivots are positive

$$\begin{bmatrix} a&-1&0\\ -1&3&1\\ 0&1&2 \end{bmatrix} \xrightarrow[\text{with } a>0]{\substack{\underset{[(a)2]}{\tau}\\ \underset{[(1)1+2]}{\tau}}} \begin{bmatrix} a&0&0\\ -1&(3a-1)&1\\ 0&a&2 \end{bmatrix} \xrightarrow[\text{with } 3a-1>0]{\substack{\underset{[(3a-1)3]}{\tau}\\ \underset{[(-1)2+3]}{\tau}}} \begin{bmatrix} a&0&0\\ -1&(3a-1)&0\\ 0&a&(5a-2) \end{bmatrix};$$

hence, $\quad a>0,\ (3a-1)>0,\ (5a-2)>0 \quad\Rightarrow\quad a>0,\ a>\frac{1}{3},\ a>\frac{2}{5} \quad\Rightarrow\quad \boxed{a>\frac{2}{5}}$.

□

(Final June 17/18) Exercise 3(a)

There are three rows ($m=3$). Since the first system has no solution, the rank is less than three, and since the second system has only one solution, all columns are pivot columns. Therefore $n=1$ or $n=2$, and so is the rank.

□

(Final June 17/18) Exercise 3(b)

Since there is no free column, the only solution is the zero vector: $\boldsymbol{x} = \mathbf{0}$.

□

(Final June 17/18) Exercise 3(c) Two possible examples are $\begin{bmatrix} 0 \\ a \\ 0 \end{bmatrix}$, and $\begin{bmatrix} 0 & b \\ a & 0 \\ 0 & c \end{bmatrix}$; where $a, c \neq 0$; but also any matrix obtained by elementary column operations from those two examples.

□

(Final June 17/18) Exercise 3(d)

The rank is the maximun number of column vectors of $\mathbf{A}$ that we can take, keeping a linearly independent the set; that is, in such way that the only linear combination of those vectors

$$a_1 \cdot \boldsymbol{column}_1 + a_2 \cdot \boldsymbol{column}_2 + \cdots + a_p \cdot \boldsymbol{column}_p$$

that equals the zero vector is when all parameters are equal to zero. But in this definition the order of the columns of $\mathbf{A}$ is irrelevant.

□

(Final June 17/18) Short questions set 1(a)

$$\left[\begin{array}{cc|c} 1 & a & -b_1 \\ -1 & 1 & -b_2 \end{array}\right] \xrightarrow{[(-a)\mathbf{1}+\mathbf{2}]^{\tau}} \left[\begin{array}{cc|c} 1 & 0 & -b_1 \\ -1 & 1+a & -b_2 \end{array}\right] \xrightarrow[\text{When } a\neq -1]{[(\frac{1}{1+a})\mathbf{2}+\mathbf{1}]^{\tau}} \left[\begin{array}{cc|c} 1 & 0 & -b_1 \\ 0 & 1+a & -b_2 \end{array}\right]$$

The system is solvable in two cases: $\boxed{\begin{cases} \text{If} \quad a \neq -1 \\ \text{If} \quad a = -1 \quad \text{and} \quad b_2 = -b_1 \end{cases}}$.

□

(Final June 17/18) Short questions set 1(b) When $\boxed{b_1 = b_2 = 0}$ (when is homogeneous) and simultaneosly $\boxed{a = -1}$ (the rank is less than two).

□

(Final June 17/18) Short questions set 1(c) When both vectors are orthogonal one to each other: $\boxed{a = 1}$.

□

(Final June 17/18) Short questions set 1(d) Using the trace, the determinant, and the fact that *both eigenvalues should be the same number*: $\begin{cases} 2\lambda = 2 \\ \lambda^2 = 1 + a \end{cases} \Rightarrow \begin{cases} \lambda = 1 \\ 1 = 1 + a \end{cases} \Rightarrow \boxed{a = 0}$.

$\boxed{\text{In this case } \mathbf{A} \text{ is not diagonalizable}}$ since it is rank 1.

□

(Final June 17/18) Short questions set 2(a) True. When the determinant is non-zero the matrix is non-singular. Hence, the matrix is full rank. . . with 4 pivots in this case.

□

(Final June 17/18) Short questions set 2(b) True. Since there is a basis of eigenvectors for $\mathbb{R}^n$, the matrix is diagonalizable, so

$$\mathbf{A} = \mathbf{SDS}^{-1} = \mathbf{IDI} = \mathbf{D},$$

where $\mathbf{D}$ is a diagonal matrix with the eigenvalues of $\mathbf{A}$ on the main diagonal.

□

(Final June 17/18) Short questions set 2(c) False. If $\boldsymbol{v}$ is an eigenvector, $a\boldsymbol{v}$ is another one. If $a \neq 1$ both vectors are distinct but dependent.

□

(Final June 17/18) Short questions set 2(d) False. For example de $n \times n$ identity matrix (with $n > 1$).

□

(Final June 17/18) Short questions set 2(e) True. That -3 is an eigenvalue means that the nullspace of $(\mathbf{A}+3\mathbf{I})$ is nontrivial (dimension > 0) so, as $\dim(\mathcal{N}\left(\mathbf{A}\right))+\text{rg}\left(\mathbf{A}\right)=n$, one must have $\text{rg}\left(\mathbf{A}\right)<n$: there must be vectors $\boldsymbol{v}$ not in the range.

□

(Final June 17/18) Short questions set 2(f) Verdadero: Una base ortonormal de $\mathcal{V}$ la forman las dos primeras columnas de la matriz identidad. Si tomamos la matriz $\underset{4\times 2}{\mathbf{Q}}$ cuyas columnas son dicha base ortonormal, tenemos que**True:** The two first columns of $\mathbf{I}_4$ form an orthogonal basis for $\mathcal{V}$. If we consider $\underset{4\times 2}{\mathbf{Q}}=[\boldsymbol{e}_1\;\boldsymbol{e}_2]$, we get

$$\mathbf{P}=\mathbf{Q}\left(\mathbf{Q}^\intercal\mathbf{Q}\right)^{-1}\mathbf{Q}^\intercal=\mathbf{Q}\mathbf{Q}^\intercal=\begin{bmatrix}1&0\\0&1\\0&0\\0&0\end{bmatrix}\begin{bmatrix}1&0&0&0\\0&1&0&0\end{bmatrix}=\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix}.$$

□

(Final May 17/18) Exercise 1(a)
$\mathbf{A}\boldsymbol{x}=c_1\mathbf{A}\boldsymbol{v}_1+c_2\mathbf{A}\boldsymbol{v}_2=2c_1\cdot\boldsymbol{v}_1+5c_2\cdot\boldsymbol{v}_2.$

□

(Final May 17/18) Exercise 1(b) Since the that eigenvectors of a symmetric matrix corresponding to different eigenvalues are orthogonal, and hence $\left[\boldsymbol{v}_1\right]^\intercal\left[\boldsymbol{v}_2\right]=0$; therefore

$$\begin{aligned}\boldsymbol{x}\mathbf{A}\boldsymbol{x}=&(c_1\boldsymbol{v}_1+c_2\boldsymbol{v}_2)\cdot(2c_1\boldsymbol{v}_1+5c_2\boldsymbol{v}_2)\\=&2c_1^2\cdot\left[\boldsymbol{v}_1\right]^\intercal\left[\boldsymbol{v}_1\right]+5c_2\cdot\left[\boldsymbol{v}_2\right]^\intercal\left[\boldsymbol{v}_2\right]\\=&2c_1^2\cdot\|\boldsymbol{v}_1\|^2+5c_2^2\cdot\|\boldsymbol{v}_2\|^2\end{aligned}$$

□

(Final May 17/18) Exercise 1(c) Since $\left[\boldsymbol{v}_i\right]^\intercal\left[\boldsymbol{v}_i\right]=\|\boldsymbol{v}_i\|^2>0$ and $c_i^2>0$ unless $c_i=0$, we conclude that

$$\boldsymbol{x}\mathbf{A}\boldsymbol{x}\;=\;2c_1^2\cdot\|\boldsymbol{v}_1\|^2+5c_2^2\cdot\|\boldsymbol{v}_2\|^2\;>0$$

unless $c_1=c_2=0$, i.e., $\boldsymbol{x}=\mathbf{0}$.

□

(Final May 17/18) Exercise 1(d) We have

$$\mathbf{B}\boldsymbol{v}_1=\left(2\cdot\left[\boldsymbol{v}_1\right]\left[\boldsymbol{v}_1\right]^\intercal+5\cdot\left[\boldsymbol{v}_2\right]\left[\boldsymbol{v}_2\right]^\intercal\right)\boldsymbol{v}_1=2\left[\boldsymbol{v}_1\right]\left[\boldsymbol{v}_1\right]^\intercal\boldsymbol{v}_1+5\left[\boldsymbol{v}_2\right]\left[\boldsymbol{v}_2\right]^\intercal\boldsymbol{v}_1=2\left[\boldsymbol{v}_1\right](1)+5\left[\boldsymbol{v}_2\right](0)=2\boldsymbol{v}_1.$$

because $\left[\boldsymbol{v}_1\right]^\intercal\left[\boldsymbol{v}_1\right]=\|\boldsymbol{v}_1\|^2=1$, and $\left[\boldsymbol{v}_i\right]^\intercal\left[\boldsymbol{v}_j\right]=0$ for $i\neq j$. Thus $\boldsymbol{v}_1$ is an eigenvector of $\mathbf{B}$ with eigenvalue $\lambda_1=2$. Similarly, we can show that $\mathbf{B}\boldsymbol{v}_2=5\boldsymbol{v}_2$:

$$\mathbf{B}\boldsymbol{v}_2=\left(2\cdot\left[\boldsymbol{v}_1\right]\left[\boldsymbol{v}_1\right]^\intercal+5\cdot\left[\boldsymbol{v}_2\right]\left[\boldsymbol{v}_2\right]^\intercal\right)\boldsymbol{v}_2=2\left[\boldsymbol{v}_1\right]\left[\boldsymbol{v}_1\right]^\intercal\boldsymbol{v}_2+5\left[\boldsymbol{v}_2\right]\left[\boldsymbol{v}_2\right]^\intercal\boldsymbol{v}_2=2\left[\boldsymbol{v}_1\right](0)+5\left[\boldsymbol{v}_2\right](1)=5\boldsymbol{v}_2.$$

□

(Final May 17/18) Exercise 1(e) Puesto que ambas $\mathbf{A}$ y $\mathbf{B}$ tienen la misma diagonalización Since both $\mathbf{A}$ and $\mathbf{B}$ have diagonalization

$$\begin{bmatrix}\boldsymbol{v}_1&\boldsymbol{v}_2\end{bmatrix}\begin{bmatrix}2&\\&5\end{bmatrix}\begin{bmatrix}\boldsymbol{v}_1&\boldsymbol{v}_2\end{bmatrix}^{-1},$$

they are the same matrix.

□

(Final May 17/18) Exercise 2(a)
Since $\mathbf{A}$ is a 3 by 3 matrix with 3 non-zero eigenvalues, $\mathbf{A}$ is full rank. Hence, the complete solution to $\mathbf{A}\boldsymbol{x}=\mathbf{0}$ is $\boldsymbol{x}=\mathbf{0}$.

□

(Final May 17/18) Exercise 2(b)
The eigenspace corresponding to $\lambda=1$ is the set of all linear combinations of $\boldsymbol{v}_2$ and $\boldsymbol{v}_3$:

$$\mathcal{E}_{\lambda=1}=\left\{\boldsymbol{x}\in\mathbb{R}^3\quad\text{such that}\quad\boldsymbol{x}=a\begin{pmatrix}1\\2\\0\end{pmatrix}+b\begin{pmatrix}0\\0\\2\end{pmatrix},\quad\forall a,b\in\mathbb{R}\right\}.$$

Applying gaussian elimination we find such a system of equations:

$$\left[\begin{array}{ccc} x & y & z \\ \hline 0 & 0 & 0 \\ \hline 1 & 2 & 0 \\ 0 & 0 & 2 \end{array}\right] \xrightarrow{[(-2)\overset{\tau}{1}+2]} \left[\begin{array}{ccc} x & y-2x & z \\ \hline 0 & 0 & 0 \\ \hline 1 & 0 & 0 \\ 0 & 0 & 2 \end{array}\right] \quad \Rightarrow \mathcal{E}_{\lambda=1} = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \quad \text{such that} \quad \{-2x+y=0 \right\}.$$

$\square$

(Final May 17/18) Exercise 2(c)

A is symmetric if, and only if, its eigenspaces are orthogonal, but

$$\begin{pmatrix} 1 & 1 & 0 \end{pmatrix} \begin{bmatrix} 1 & 0 \\ 2 & 0 \\ 0 & 2 \end{bmatrix} = \begin{pmatrix} 3 & 0 \end{pmatrix} \neq \mathbf{0}.$$

Since $\mathcal{E}_{\lambda=1}$ is not perpendicular to $\mathcal{E}_{\lambda=2}$, $\boxed{\textbf{A} \text{ is not symmetric}}$.

But, since $\boldsymbol{v}_1$, $\boldsymbol{v}_2$ and $\boldsymbol{v}_3$ are linearly independent (because $\mathbf{S} = \begin{bmatrix} \boldsymbol{v}_1 & \boldsymbol{v}_2 & \boldsymbol{v}_3 \end{bmatrix}$

$$\mathbf{S} = \begin{bmatrix} \boldsymbol{v}_1 & \boldsymbol{v}_2 & \boldsymbol{v}_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \xrightarrow{[(-1)\overset{\tau}{1}+2]} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}; \qquad \text{it is fullrank});$$

matrix $\boxed{\textbf{A} \text{ is diagonalizable}}$.

$\square$

(Final May 17/18) Exercise 2(d)

Since $\mathbb{R}^3$ has dimension 3, and $\boldsymbol{v}_1$, $\boldsymbol{v}_2$, and $\boldsymbol{v}_3$ are three linear independent vectors in $\mathbb{R}^3$ (see part (c)), B is a basis of $\mathbb{R}^3$.

By gaussian elimination we get

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & -1 \\ 1 & 2 & 0 & -0 \\ 0 & 0 & 2 & -1 \\ \hline 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \xrightarrow[{[(-1)\overset{\tau}{2}+1]}]{[(-1)\overset{\tau}{1}+2]} \left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & -1 \\ \hline 2 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \xrightarrow[{[(1/2)\overset{\tau}{3}+4]}]{[(1)\overset{\tau}{1}+4]} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ \hline 2 & -1 & 0 & 2 \\ -1 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1/2 \end{array}\right]$$

Hence, the coordinates of $\begin{pmatrix} 1 & 0 & 1 \end{pmatrix}$ with respect to B are $x = 2$, $y = -1$ and $z = 1/2$; since

$$\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = 2\boldsymbol{v}_1 - \boldsymbol{v}_2 + \frac{1}{2}\boldsymbol{v}_3.$$

$\square$

(Final May 17/18) Exercise 2(e)

$$\mathbf{A}^3 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \mathbf{A}^3(2\boldsymbol{v}_1 - \boldsymbol{v}_2 + \frac{1}{2}\boldsymbol{v}_3) = 2 \cdot \mathbf{A}^3\boldsymbol{v}_1 - \mathbf{A}^3\boldsymbol{v}_2 + \frac{1}{2} \cdot \mathbf{A}^3\boldsymbol{v}_3 = 2 \cdot 2^3\boldsymbol{v}_1 - \boldsymbol{v}_2 + \frac{1}{2}\boldsymbol{v}_3;$$

that is

$$\mathbf{A}^3 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = 16 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 15 \\ 14 \\ 1 \end{pmatrix}.$$

$\square$

(Final May 17/18) Exercise 3(a)

Since

$$\mathbf{H}^\mathsf{T}\mathbf{H} = \begin{bmatrix} 2 & & \\ & 2 & \\ & & 9 \end{bmatrix}$$

columns of $\mathbf{H}$ are perpendicular and therefore **they are an orthogonal basis** of $\mathbb{R}^4$; but since they are nor unit vectors **they are NOT an orthonormal basis** of $\mathbb{R}^4$ (the two first columns have norm $\sqrt{2}$ and the last one has norm 3).

□

(Final May 17/18) Exercise 3(b)

Since the two first columns have norm $\sqrt{2}$ and the last one has norm 3, then

$$\mathbf{Q} = \mathbf{H}\mathbf{D} = \begin{bmatrix} 1 & 1 & \\ -1 & 1 & \\ & & 3 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & & \\ & \frac{1}{\sqrt{2}} & \\ & & \frac{1}{3} \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & \\ -1/\sqrt{2} & 1/\sqrt{2} & \\ & & 1 \end{bmatrix}.$$

Hence, $\mathbf{Q}^{-1} = (\mathbf{H}\mathbf{D})^{-1} = \mathbf{D}^{-1}\mathbf{H}^{-1}$, and then, multiplying by $\mathbf{D}$ we get: $\mathbf{D}\mathbf{Q}^{-1} = \mathbf{H}^{-1}$. But, since $\mathbf{Q}$ is orthogonal, $\mathbf{Q}^{-1} = \mathbf{Q}^{\intercal}$, so finally we get $\mathbf{H}^{-1} = \mathbf{D}\mathbf{Q}^{\intercal}$:

$$\mathbf{H}^{-1} = \begin{bmatrix} \frac{1}{\sqrt{2}} & & \\ & \frac{1}{\sqrt{2}} & \\ & & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} & \\ 1/\sqrt{2} & 1/\sqrt{2} & \\ & & 1 \end{bmatrix} = \begin{bmatrix} 1/2 & -1/2 & \\ 1/2 & 1/2 & \\ & & 1/3 \end{bmatrix}.$$

□

(Final May 17/18) Exercise 3(c)

$$\left[\begin{array}{ccc} x & y & z \\ \hline 0 & 0 & 0 \\ \hline 1 & -1 & 0 \end{array}\right] \xrightarrow{\underset{[(1)\mathbf{1}+\mathbf{2}]}{\tau}} \left[\begin{array}{ccc} x & \color{red}{(y+x)} & \color{red}{z} \\ \hline 0 & \color{red}{0} & \color{red}{0} \\ \hline 1 & \color{blue}{0} & \color{blue}{0} \end{array}\right] \quad \Rightarrow \quad \begin{cases} x+y & =0 \\ & z=0 \end{cases}.$$

□

(Final May 17/18) Exercise 3(d)

Projection matrix $\mathbf{P}$ is $\mathbf{A}(\mathbf{A}^{\intercal}\mathbf{A})^{-1}\mathbf{A}^{\intercal}$, where $\mathbf{A} = \begin{bmatrix} \mathbf{H}_{|1} & \mathbf{H}_{|3} \end{bmatrix}$; therefore

$$\mathbf{P} = \begin{bmatrix} 1 & 0 \\ -1 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \\ & \frac{1}{9} \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 1/2 & 0 \\ -1/2 & 0 \\ 0 & 1/3 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 1/2 & -1/2 & 0 \\ -1/2 & 1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

□

(Final May 17/18) Short questions set 1(a)

For example:

$$C_{22} = \begin{vmatrix} 1 & 0 & 1 \\ 0 & 2 & 3 \\ 0 & 0 & 2 \end{vmatrix} = 4.$$

□

(Final May 17/18) Short questions set 1(b)

If we consider gaussian elimination by rows, we must sustract two times the first row from the second one. And then two times the third row from the forth $\mathbf{I}_{\underset{[(-2)\mathbf{3}+\mathbf{4}]}{\tau}} \mathbf{I}_{\underset{[(-2)\mathbf{2}+\mathbf{3}]}{\tau}} \mathbf{A} = \mathbf{U}$:

$$\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & -2 & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 2 & 3 \\ 0 & 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

If we consider gaussian elimination by columns, we need three elementary transformations: $\mathbf{A}\mathbf{I}_{\underset{[(-2)\mathbf{1}+\mathbf{2}]}{\tau}} \mathbf{I}_{\underset{[(-1)\mathbf{1}+\mathbf{4}]}{\tau}} \mathbf{I}_{\underset{[(-1)\mathbf{3}+\mathbf{4}]}{\tau}} = \mathbf{L}$:

$$\begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 2 & 3 \\ 0 & 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & -2 & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & -1 \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & -1 \\ & & & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix}.$$

□

(Final May 17/18) Short questions set 1(c)

$$\left[\begin{array}{cccc} 1 & 2 & 0 & 1\\ 0 & 0 & 1 & 1\\ 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 0\\ \hline 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow[\substack{[(-2)\mathbf{1}+\mathbf{2}]\\(-1)\mathbf{1}+\mathbf{4}}]{\boldsymbol{\tau}} \left[\begin{array}{cccc} 1 & 0 & 0 & 0\\ 0 & 0 & 1 & 1\\ 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 0\\ \hline 1 & -2 & 0 & -1\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow[{[(-1)\mathbf{3}+\mathbf{4}]}]{\boldsymbol{\tau}} \left[\begin{array}{cccc} 1 & 0 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 0\\ \hline 1 & -2 & 0 & -1\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & -1\\ 0 & 0 & 0 & 1 \end{array}\right]$$

Hence, $\dim \mathcal{N}\left(\mathbf{A}\right) = 1$ and a basis for this subspace is

$$\text{Basis for } \mathcal{N}\left(\mathbf{A}\right) = \left\{ \begin{pmatrix} -2\\ 1\\ 0\\ 0 \end{pmatrix} \right\}.$$

□

(Final May 17/18) Short questions set 1(d)

$\mathbf{A}$ is diagonalizable if the eigenspace corresponding to the double eigenvalue $\lambda = 2$ has dimansion 2: $\dim(\mathcal{E}_{\lambda=2}) = \dim(\mathcal{N}(\mathbf{A} - 2\mathbf{I}) = 2$.

$$\left[\mathbf{A} - 2\mathbf{I}\right] = \begin{bmatrix} -1 & 2 & 0 & 1\\ 0 & -2 & 1 & 1\\ 0 & 0 & 0 & 3\\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow[\substack{[(2)\mathbf{1}+\mathbf{2}]\\(2)\mathbf{3}+\mathbf{2}}]{\boldsymbol{\tau}} \begin{bmatrix} -1 & 0 & 0 & 1\\ 0 & 0 & 1 & 1\\ 0 & 0 & 0 & 3\\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow[\substack{[(1)\mathbf{1}+\mathbf{4}]\\(-1)\mathbf{3}+\mathbf{4}}]{\boldsymbol{\tau}} \begin{bmatrix} -1 & 0 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 3\\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Since $(\mathbf{A} - 2\mathbf{I})$ has rank 3, then $\dim(\mathcal{E}_{\lambda=2}) = \dim(\mathcal{N}(\mathbf{A} - 2\mathbf{I}) = 1$ and therefore $\boxed{\mathbf{A} \text{ is NOT diagonalizable}}$.

□

(Final May 17/18) Short questions set 1(e)

$$\det\left(\mathbf{A} - \mathbf{A}^{\mathsf{T}}\right) = \begin{vmatrix} 0 & 2 & 0 & 1\\ -2 & 0 & 1 & 1\\ 0 & -1 & 0 & 3\\ -1 & -1 & -3 & 0 \end{vmatrix} = \begin{vmatrix} 0 & 0 & 0 & 1\\ -2 & -2 & 1 & 1\\ 0 & -7 & 0 & 3\\ -1 & -1 & -3 & 0 \end{vmatrix} = - \begin{vmatrix} -2 & -2 & 1\\ 0 & -7 & 0\\ -1 & -1 & -3 \end{vmatrix} = - \begin{vmatrix} 0 & 0 & 1\\ 0 & -7 & 0\\ -7 & -7 & -3 \end{vmatrix} = 49.$$

□

(Final May 17/18) Short questions set 1(f)

Since rows (0 0 1 1) and (0 0 2 3) are linearly independent, that subspace consist in all vectors in $\mathbb{R}^4$ whose two firts components are zero, that is, all vector with the form: (0 0 a b).

Hence, it is enough if we find a vector (0 0 a b) perpendicular to (0 0 1 1). For example (0 0 1 -1).

Now we just only need to normalize those two vectors. Since both have length $\sqrt{2}$, an orthonormal basis is

$$\left\{ \frac{1}{\sqrt{2}}\begin{pmatrix} 0\\ 0\\ 1\\ 1 \end{pmatrix}; \frac{1}{\sqrt{2}}\begin{pmatrix} 0\\ 0\\ 1\\ -1 \end{pmatrix} \right\}.$$

□

(Final May 17/18) Short questions set 2(a)

$$|\mathbf{B}| = \det\left[(2\mathbf{A}_{|1}), (\mathbf{A}_{|2} + 7\mathbf{A}_{|1}), (-\mathbf{A}_{|3})\right] = 2\det\left[\mathbf{A}_{|1}, (\mathbf{A}_{|2} + 7\mathbf{A}_{|1}), (-\mathbf{A}_{|3})\right]$$

$$= 2\det\left[\mathbf{A}_{|1}, \mathbf{A}_{|2}, (-\mathbf{A}_{|3})\right] = -2\det\left[\mathbf{A}_{|1}, \mathbf{A}_{|2}, \mathbf{A}_{|3}\right] = -2|\mathbf{A}| = -20.$$

□

(Final May 17/18) Short questions set 2(b)

$$\det\left[(\mathbf{A}^{-1}\mathbf{B}^\intercal)^{-1}\right] = \det\left[(\mathbf{B}^\intercal)^{-1}\mathbf{A}\right] = \det\left[\mathbf{B}^{-1}\right]\cdot\det\left[\mathbf{A}\right] = \frac{(10)}{(-20)} = \frac{-1}{2}.$$

□

(Final May 17/18) Short questions set 3(a)

Let $\mathbf{A}$ be the corresponding symmetric matrix such that $f(x,y) = f(\boldsymbol{x}) = \boldsymbol{x}\mathbf{A}\boldsymbol{x}$; by gaussian elimination (if $a \neq 0$) we get

$$\mathbf{A} = \begin{bmatrix} a & 3 \\ 3 & a \end{bmatrix} \to \begin{bmatrix} a & 0 \\ 3 & a - \frac{9}{a} \end{bmatrix}; \qquad (\text{if } a \neq 0)$$

On the one hand, when $a = 0$ the trace of $\mathbf{A}$ is zero but the determinant is not, therefore one eigenvalue is positive and the other one is negative. When $a > 0$ at least one eigenvalue is positive, and when $a > 0$ at least one eigenvalue is negative.

On the other hand, an eigenvalue is zero when $a - \frac{9}{a} = 0$, that is, when $a^2 - 9 = 0$. In other words, one eigenvalue is zero when $a = \pm 3$

Summarizing:

$$\begin{cases} a < -3 & \text{negative definite} \\ a = -3 & \text{negative } semi\text{definite} \\ a \in (-3,\ 3) & \text{not positive nor negative} \\ a = 3 & \text{positive } semi\text{definite} \\ a > 3 & \text{positive definite} \end{cases}$$

□

(Final May 17/18) Short questions set 3(b)

The set of solutions is the set of points that satisfy $x = y$. Hence, $\mathbf{A}$ must be singular, and therefore, a must be equal to 3 or -3. We only need to guess which value is the right one.

The set of points such that $x = y$, is the set of multiples of $(1,\ 1)$. So $\mathbf{A}\begin{pmatrix}1\\1\end{pmatrix} = \mathbf{0}$, and therefore $\boxed{a \text{ must be equal to } -3}$ so $\mathbf{A} = \begin{bmatrix} -3 & 3 \\ 3 & -3 \end{bmatrix}$.

□

(Final July 16/17) Exercise 1(a)

Matrix $\mathbf{A}$ has 4 columns, but one of them is free (since $\dim \mathcal{N}\left(\mathbf{A}\right) = 1$), hence, there are three pivot columns. Since $\mathbf{A}$ is full row rank, there are three pivot rows (no free rows in this matrix). So: $\underset{3\times 4}{\mathbf{A}}$.

□

(Final July 16/17) Exercise 1(b)

We only need to find three row vectors orthogonal to $\mathcal{N}\left(\mathbf{A}\right)$:

$$\left[\begin{array}{cccc} -1 & 2 & -3 & 1 \\ \hline 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow[\substack{(-3)\mathbf{1}+\mathbf{3} \\ (1)\mathbf{1}+\mathbf{4}}]{\substack{\boldsymbol{\tau} \\ [(2)\mathbf{1}+\mathbf{2}]}} \left[\begin{array}{cccc} -1 & 0 & 0 & 0 \\ \hline 1 & 2 & -3 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

Hence, the matrix $\mathbf{A} = \begin{bmatrix} 2 & 1 & 0 & 0 \\ -3 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$ fulfills the conditions; but note that applying elementary transformations on the rows we can get another example: $\mathbf{B} = \mathbf{E}\mathbf{A}$, where $\mathbf{E}$ is an invertible matrix.

□

(Final July 16/17) Exercise 1(c)

The system is solvable for any $\boldsymbol{b} \in \mathbb{R}^3$ since $\underset{3\times 4}{\mathbf{A}}$ is a full row rank.

□

(Final July 16/17) Exercise 1(d)

For any matrix $\mathbf{A}$, the null space $\mathcal{N}\left(\mathbf{A}\right)$ is the orthogonal complement to the row space $\mathcal{C}\left(\mathbf{A}^\intercal\right)$. Hence, any solution to $\mathbf{A}\boldsymbol{x} = \mathbf{0}$, that is, any non-zero multiple of $\left(-1,\quad 2,\quad -3,\quad 1\right)$ is a vector in $\mathbb{R}^n$ that is not in the row space of $\mathbf{A}$.

□

(Final July 16/17) Exercise 2(a)

Since we need to solve a linear system in part b), it is a good idea work with the augmented matrix $\left[\boldsymbol{u}_1,\, \boldsymbol{u}_2,\, \boldsymbol{u}_3,\, \boldsymbol{u}_4 \,|\, -\boldsymbol{b}\right] \equiv \left[\mathbf{B} \,|\, -\boldsymbol{b}\right]$, where $\boldsymbol{b}$ is the right hand side vector of linear system in part b) and $\mathbf{B}$ is its coefficient matrix:

$$\left[\begin{array}{cccc|c} -1 & 1 & 0 & 0 & -1 \\ 1 & -2 & 0 & 0 & -1 \\ 0 & 0 & -2 & -2 & -0 \\ 0 & 0 & 4 & -3 & -1 \\ \hline 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right] \xrightarrow{\substack{\overset{\tau}{[(1)\mathbf{1}+\mathbf{2}]} \\ \overset{\tau}{[(-1)\mathbf{3}+\mathbf{4}]}}} \left[\begin{array}{cccc|c} -1 & 0 & 0 & 0 & -1 \\ 1 & -1 & 0 & 0 & -1 \\ 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 4 & -7 & -1 \\ \hline 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right] \xrightarrow{\substack{\overset{\tau}{[(-1)\mathbf{1}]} \\ (-1)\mathbf{2} \\ (-1/2)\mathbf{3} \\ (-1/7)\mathbf{4}}} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & -1 \\ -1 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -2 & 1 & -1 \\ \hline -1 & -1 & -0 & -0 & 0 \\ -0 & -1 & -0 & -0 & 0 \\ -0 & -0 & -1/2 & 1/7 & 0 \\ -0 & -0 & -0 & -1/7 & 0 \end{array}\right]$$

$$\xrightarrow{\substack{\overset{\tau}{[(1)\mathbf{2}+\mathbf{1}]} \\ \overset{\tau}{[(2)\mathbf{4}+\mathbf{3}]}}} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 \\ \hline -2 & -1 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 & 0 \\ 0 & 0 & -3/14 & 1/7 & 0 \\ 0 & 0 & -2/7 & -1/7 & 0 \end{array}\right] \xrightarrow{\substack{\overset{\tau}{[(1)\mathbf{1}+\mathbf{5}]} \\ \overset{\tau}{[(1)\mathbf{2}+\mathbf{5}]} \\ \overset{\tau}{[(1)\mathbf{3}+\mathbf{5}]}}} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ \hline -2 & -1 & 0 & 0 & -3 \\ -1 & -1 & 0 & 0 & -2 \\ 0 & 0 & -3/14 & 1/7 & 1/7 \\ 0 & 0 & -2/7 & -1/7 & -1/7 \end{array}\right]$$

since all columns in $\mathbf{B}$ are pivot columns, all vectors in B are linearly independent. Since vectors in B belong to $\mathbb{R}^4$ (a four dimensional vector space), B must be a basis for $\mathbb{R}^4$.

□

(Final July 16/17) Exercise 2(b)

The coordinates are $x = -3,\ y = -2,\ z = \frac{1}{7},\ w = \frac{-1}{7}$; in other words,

$$-3\boldsymbol{u}_1 - 2\boldsymbol{u}_2 + \frac{1}{7}\boldsymbol{u}_1 - \frac{1}{7}\boldsymbol{u}_1$$

as you can check:

$$-3\begin{pmatrix}-1\\1\\0\\0\end{pmatrix} - 2\begin{pmatrix}1\\-2\\0\\0\end{pmatrix} + \frac{1}{7}\begin{pmatrix}0\\0\\-2\\4\end{pmatrix} - \frac{1}{7}\begin{pmatrix}0\\0\\-2\\-3\end{pmatrix} = \begin{pmatrix}1\\1\\0\\1\end{pmatrix}.$$

□

(Final July 16/17) Exercise 2(c)

Using the information above we have

$$\mathbf{AB} = \mathbf{A}\begin{bmatrix}\boldsymbol{u}_1 & \boldsymbol{u}_2 & \boldsymbol{u}_3 & \boldsymbol{u}_4\end{bmatrix} = \begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&2&0\\0&0&0&-1\end{bmatrix}.$$

Since $\mathbf{AB}$ and $\mathbf{B}$ are full rank matrices, then $\mathbf{A}$ must be full rank. Noticing that column 3 of $\mathbf{AB}$ is twice the third column of $\mathbf{I}$ and column 4 of $\mathbf{AB}$ is -1 times the fourth column of $\mathbf{I}$, it is easy to see that $\mathbf{A}\begin{bmatrix}\boldsymbol{u}_1 & \boldsymbol{u}_2 & \frac{1}{2}\boldsymbol{u}_3 & -\boldsymbol{u}_4\end{bmatrix} = \mathbf{I}$. Therefore

$$\mathbf{A}^{-1} = \begin{bmatrix}\boldsymbol{u}_1 & \boldsymbol{u}_2 & \frac{1}{2}\boldsymbol{u}_3 & -\boldsymbol{u}_4\end{bmatrix} = \begin{bmatrix}-1&1&0&0\\1&-2&0&0\\0&0&-1&2\\0&0&2&3\end{bmatrix}.$$

□

(Final July 16/17) Exercise 2(d)

On the one hand

$$f(\boldsymbol{u}_1) = \boldsymbol{u}_1\mathbf{A}\boldsymbol{u}_1 = \boldsymbol{u}_1\Big(\mathbf{A}\boldsymbol{u}_1\Big) = \begin{pmatrix}-1 & 1 & 0 & 0\end{pmatrix}\begin{pmatrix}1\\0\\0\\0\end{pmatrix} = -1 < 0$$

on the other hand

$$f(\boldsymbol{u}_4) = \boldsymbol{u}_4 \mathbf{A} \boldsymbol{u}_4 = \boldsymbol{u}_4 \Big(\mathbf{A} \boldsymbol{u}_4\Big) = \begin{pmatrix} 0 & 0 & -2 & -3 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ -1 \end{pmatrix} = 3 > 0$$

Hence, this quadratic form is neither positive definited nor negative definited, since it can take both positive and negative values.

□

(Final July 16/17) Exercise 3(a)

When computing $\mathbf{B}\boldsymbol{v}$ we get $4\boldsymbol{v}$. Matrix $\mathbf{B}$ is rank one symmetric with trace 4, so the eigenvalues are 4 and 0 (with algebraic multiplicity 3).

□

(Final July 16/17) Exercise 3(b)

$$\Big(\mathbf{B} + b\mathbf{I}\Big)\boldsymbol{x} = \mathbf{B}\boldsymbol{x} + b\mathbf{I}\boldsymbol{x} = \lambda\boldsymbol{x} + b\boldsymbol{x} = (\lambda + b)\boldsymbol{x},$$

so $\mathbf{A}$ has the same eigenvectors and its eigenvalues are b, b, b and $4 + b$.

□

(Final July 16/17) Exercise 3(c)

The eigenvalues of $\mathbf{A}$ are 2, 2, 2, and $2 + 4 = 6$ so the determinant is $2 * 2 * 2 * 6 = 48$.

□

(Final July 16/17) Exercise 3(d)

We need $b > 0$.

□

(Final July 16/17) Exercise 3(e)

Since $\mathbf{B}^2 = 4\mathbf{B}$ and $\mathbf{I} = \mathbf{A}\mathbf{A}^{-1} = (\mathbf{B} + \mathbf{I})(\mathbf{I} + c\mathbf{B}) = \mathbf{B} + 4c\mathbf{B} + \mathbf{I} + c\mathbf{B} = \mathbf{I} + (1 + 5c)\mathbf{B}$ so $(\mathbf{I} + c\mathbf{B})$ is the inverse of $\mathbf{A}$ if $(1 + 5c) = 0$. Therefore $c = -1/5$.

□

(Final July 16/17) Short questions set 1(a)

False: $\begin{bmatrix} \boldsymbol{u} & \boldsymbol{v} & \boldsymbol{w} \end{bmatrix} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = \boldsymbol{u} - \boldsymbol{v}.$ Therefore, its coordinates are $(1, -1, 0)$.

□

(Final July 16/17) Short questions set 1(b)

True: since it is a linear combination of vectors from basis B.

□

(Final July 16/17) Short questions set 1(c)

It is *symmetric* since $\mathbf{P}^\intercal = \Big(\mathbf{X}\mathbf{X}^\intercal\Big)^\intercal = (\mathbf{X}^\intercal)^\intercal\mathbf{X}^\intercal = \mathbf{X}\mathbf{X}^\intercal = \mathbf{P}$.

Since B is an orthonormal basis then $\mathbf{X}^\intercal\mathbf{X} = \underset{3\times 3}{\mathbf{I}}$. Hence $\mathbf{P}$ is *idemppotent* since $\mathbf{P}\mathbf{P} = \mathbf{X}\mathbf{X}^\intercal\mathbf{X}\mathbf{X}^\intercal = \mathbf{X}\mathbf{I}\mathbf{X}^\intercal = \mathbf{X}\mathbf{X}^\intercal = \mathbf{P}$,

□

(Final July 16/17) Short questions set 1(d)

We need to prove that $(\mathbf{P}\boldsymbol{y}) \cdot (\boldsymbol{y} - \mathbf{P}\boldsymbol{y}) = (\boldsymbol{y}\mathbf{P}^\intercal) \cdot (\boldsymbol{y} - \mathbf{P}\boldsymbol{y})$ is zero. Let's see...

$$(\boldsymbol{y}\mathbf{P}^\intercal) \cdot (\boldsymbol{y} - \mathbf{P}\boldsymbol{y}) = \boldsymbol{y}\mathbf{P}^\intercal\boldsymbol{y} - \boldsymbol{y}\mathbf{P}^\intercal\mathbf{P}\boldsymbol{y} = \boldsymbol{y}\mathbf{P}\boldsymbol{y} - \boldsymbol{y}\mathbf{P}\boldsymbol{y} = 0$$

where $\mathbf{P}^\intercal = \mathbf{P}$ and $\mathbf{P}^\intercal\mathbf{P} = \mathbf{P}\mathbf{P} = \mathbf{P}$.

□

(Final July 16/17) Short questions set 2(a)

Since we are not asked to compute $\mathbf{A}^{-1}$, it is enough to get a triangular matrix $\mathbf{L}$ using elementary transformations. Analyzing $\mathbf{L}$ we can then find the determinant

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & a & 0 & -1 \end{bmatrix} \xrightarrow[{[(-1)\mathbf{1}+\mathbf{3}]}]{\boldsymbol{\tau}} \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & a & 0 & -1 \end{bmatrix} \xrightarrow[{[(-1)\mathbf{2}+\mathbf{3}]}]{\boldsymbol{\tau}} \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & a & -a & -1 \end{bmatrix} \xrightarrow[{[(-1)\mathbf{3}+\mathbf{4}]}]{\boldsymbol{\tau}} \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & a & -a & a-1 \end{bmatrix}$$

Since we have used Type I elementary transformations only, then $\det \mathbf{A} = \det \mathbf{L} = a - 1$. So there exists $\mathbf{A}^{-1}$ if and only if $a \neq 1$.

□

(Final July 16/17) Short questions set 2(b)

We need to solve

$$\det\left[(\mathbf{A}^{\intercal}\mathbf{A})^{-1}\mathbf{A}^{\intercal}\right] = \frac{\det \mathbf{A}^{\intercal}}{\det \mathbf{A}^{\intercal} \det \mathbf{A}} = \frac{\det \mathbf{A}}{(\det \mathbf{A})^2} = \frac{a-1}{(a-1)^2} = \frac{1}{(a-1)} = \frac{1}{4},$$

so $a - 1 = 4$. Therefore: $a = 5$.

□

(Final July 16/17) Short questions set 2(c)

Since $\mathbf{A}$ has rank 3 when $a = 1$, the answer is $a = 1$.

□

(Final July 16/17) Short questions set 3(a)

Let's find the roots of the characteristic polynomial:

$$\det(\mathbf{A} - \lambda \mathbf{I}) = \begin{vmatrix} -\lambda & 3 & 0 \\ a & -\lambda & b \\ 0 & 4 & -\lambda \end{vmatrix} = (-\lambda)^3 + 4b\lambda + 3a\lambda = 0$$

Zero is an eigenvalue. The other two roots come from: $-(\lambda)^2 + 4b + 3a = 0$; that is $\lambda^2 = 3a + 4b$. Those roots are real only if $3a + 4b \geq 0$. Hence, there are two cases:

- When $3a+4b = 0$ matrix **$\mathbf{A}$ is not diagonalizable**: three eigenvalues equal to zero, but $\dim \mathcal{N}\left(\mathbf{A}\right) < 3$.
- **When $3a + 4b > 0$ matrix $\mathbf{A}$ is diagonalizable** since there are three different eigenvalues: 0 and $\pm\sqrt{3a + 4b}$.

  Therefore, for any b, $\mathbf{A}$ is diagonalizable if and only if $a > \frac{-4}{3}b$.

□

(Final July 16/17) Short questions set 3(b)

Since $\pm\sqrt{3a + 4b} = \pm\sqrt{25}$, the three eigenvalues are 0, -5 and 5; hence this quadratic form $\boldsymbol{x}\mathbf{A}\boldsymbol{x}$ is neither positive definited nor negative definited.

□

(Final July 16/17) Short questions set 3(c)

Matrix $\mathbf{A}$ must be symmetric, so $a = 3$ and $b = 4$. First, we find an eigenvector with eigenvalue 5...

$$\left[\begin{array}{c} \mathbf{A} - 5\mathbf{I} \\ \hline \mathbf{I} \end{array}\right] = \left[\begin{array}{rrr} -5 & 3 & 0 \\ 3 & -5 & 4 \\ 0 & 4 & -5 \\ \hline 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right] \xrightarrow[{[(5)\mathbf{2}]}]{\substack{\boldsymbol{\tau} \\ [(3)\mathbf{1}] \\ \boldsymbol{\tau}}} \left[\begin{array}{rrr} -15 & 15 & 0 \\ 9 & -25 & 4 \\ 0 & 20 & -5 \\ \hline 3 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{array}\right] \xrightarrow{\substack{\boldsymbol{\tau} \\ [(1)\mathbf{1}+\mathbf{2}]}} \left[\begin{array}{rrr} -15 & 0 & 0 \\ 9 & -16 & 4 \\ 0 & 20 & -5 \\ \hline 3 & 3 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{array}\right] \xrightarrow{\substack{\boldsymbol{\tau} \\ [(4)\mathbf{3}] \\ (1)\mathbf{2}+\mathbf{3}}} \left[\begin{array}{rrr} -15 & 0 & 0 \\ 9 & -16 & 0 \\ 0 & 20 & 0 \\ \hline 3 & 3 & 3 \\ 0 & 5 & 5 \\ 0 & 0 & 4 \end{array}\right]$$

so $\begin{pmatrix} 3 \\ 5 \\ 4 \end{pmatrix}$ is an eigenvector with correspondant eigenvalue 5. Now, since the length is $\sqrt{3^2 + 5^2 + 4^2} = \sqrt{9 + 25 + 16} = \sqrt{50} = 5\sqrt{2}$, an answer could be: $\boldsymbol{v} = \frac{1}{5\sqrt{2}}\begin{pmatrix} 3 & 5 & 4 \end{pmatrix}$. The other answer (switching the sign) is: $\boldsymbol{v} = \frac{-1}{5\sqrt{2}}\begin{pmatrix} 3 & 5 & 4 \end{pmatrix}$.

□

(Final May 16/17) Exercise 1(a)

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & a & 1 & b \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & -1 & 1 & 1 \\ 0 & a & 1 & b-a \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 \\ 0 & a & 1 & b-a-1 \end{bmatrix}$$

They form a basis when $b - a \neq 1$.

$\square$

(Final May 16/17) Exercise 1(b)

$$\left[\begin{array}{cccc} 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ a & 0 & 1 & a \\ \hline 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{\mathbf{P}_{12}} \left[\begin{array}{cccc} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & a & 1 & a \\ \hline 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{\substack{\mathcal{T} \\ [(-1)\mathbf{1+2}] \\ (-1)\mathbf{2+4} \\ (-1)\mathbf{3+4}}} \left[\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 \\ 0 & a & 1 & -1 \\ \hline 0 & 1 & 0 & -1 \\ 1 & -1 & 0 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{\substack{\mathcal{T} \\ [(-1)\mathbf{4}] \\ (-1)\mathbf{4+3} \\ (-a)\mathbf{4+2}}} \left[\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \hline 0 & 1-a & -1 & 1 \\ 1 & a-1 & 1 & -1 \\ 0 & -a & 0 & 1 \\ 0 & a & 1 & -1 \end{array}\right]$$

$$\xrightarrow{\substack{\mathcal{T} \\ [(1)\mathbf{3+2}] \\ (-1)\mathbf{3+1}}} \left[\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \hline 1 & -a & -1 & 1 \\ 0 & a & 1 & -1 \\ 0 & -a & 0 & 1 \\ -1 & a+1 & 1 & -1 \end{array}\right] \quad \Rightarrow \quad \mathbf{A}^{-1} = \begin{bmatrix} 1 & -a & -1 & 1 \\ 0 & a & 1 & -1 \\ 0 & -a & 0 & 1 \\ -1 & a+1 & 1 & -1 \end{bmatrix}$$

$\square$

(Final May 16/17) Exercise 1(c)

$$\dim \mathcal{S} = 3; \quad \text{the set } \left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} \right\} \text{ is a basis for } \mathcal{S}.$$

$\square$

(Final May 16/17) Exercise 1(d)

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & -1 \\ 1 & 0 & 0 & -1/2 \\ 0 & 1 & 1 & -1/2 \\ 0 & 0 & 1 & 0 \\ \hline 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \xrightarrow{\substack{\mathcal{T} \\ [(-1)\mathbf{1+2}] \\ (1)\mathbf{1+4}}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 1/2 \\ 0 & 1 & 1 & -1/2 \\ 0 & 0 & 1 & 0 \\ \hline 1 & -1 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \xrightarrow{\substack{\mathcal{T} \\ [(1/2)\mathbf{2+4}]}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ \hline 1 & -1 & 0 & 1/2 \\ 0 & 1 & 0 & 1/2 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

Hence, the coordinates are $\begin{pmatrix} 1/2, & 1/2, & 0 \end{pmatrix}$, that is, $\boldsymbol{b} = \frac{1}{2}\boldsymbol{v}_1 + \frac{1}{2}\boldsymbol{v}_2 + 0\boldsymbol{v}_3$.

$\square$

(Final May 16/17) Exercise 2(a)

Suppose $\mathbf{H}\boldsymbol{v} = \lambda\boldsymbol{v}$ for some non-zero vector $\boldsymbol{v}$. Then $\mathbf{H}^2\boldsymbol{v} = \lambda^2\boldsymbol{v} = (4\mathbf{I})\boldsymbol{v} = 4\boldsymbol{v}$, so $\lambda^2 = 4$, and thus every eigenvalue of $\mathbf{H}$ is equal to either 2 or -2. The trace of $\mathbf{H}$ is 0, hence the sum of the eigenvalues of $\mathbf{H}$ is 0. We conclude that $\mathbf{H}$ has eigenvalues $\lambda = 2, 2, -2, -2$.

$\square$

(Final May 16/17) Exercise 2(b)

From $\mathbf{H}^2 = 4\mathbf{I}$ we obtain

$$\mathbf{H}\mathbf{H} = 4\mathbf{I} \quad \Rightarrow \quad \mathbf{H}\frac{1}{4}\mathbf{H} = \mathbf{I} \quad \Rightarrow \quad \mathbf{H}^{-1} = \frac{1}{4}\mathbf{H}$$

The determinant of a matrix is the product of its eigenvalues: $\det \mathbf{H} = 2 \cdot 2 \cdot (-2) \cdot (-2) = 16$.

$\square$

(Final May 16/17) Exercise 2(c)

Since $\mathbf{H}$ is symmetric and the three given eigenvectors are pairwise orthogonal, any non-zero vector perpendicular to them is automatically a fourth eigenvector. Hence, by gaussian elimination:

$$\left[\begin{array}{rrrr} 1 & 1 & 1 & -1 \\ 1 & 0 & 0 & 1 \\ 0 & -1 & 1 & 0 \\ \hline 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{\substack{\boldsymbol{\tau} \\ {[(-1)\mathbf{1}+\mathbf{2}]} \\ (-1)\mathbf{1}+\mathbf{3} \\ (1)\mathbf{1}+\mathbf{4}}} \left[\begin{array}{rrrr} 1 & 0 & 0 & 0 \\ 1 & -1 & -1 & 2 \\ 0 & -1 & 1 & 0 \\ \hline 1 & -1 & -1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{\substack{\boldsymbol{\tau} \\ {[(-1)\mathbf{2}+\mathbf{3}]} \\ (2)\mathbf{2}+\mathbf{4}}} \left[\begin{array}{rrrr} 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & -1 & 2 & -2 \\ \hline 1 & -1 & 0 & -1 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{\substack{\boldsymbol{\tau} \\ {[(1)\mathbf{3}+\mathbf{4}]}}} \left[\begin{array}{rrrr} 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & -1 & 2 & 0 \\ \hline 1 & -1 & 0 & -1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

we conclude that another eigenvector is

$$\boldsymbol{v}_4 = \begin{pmatrix} -1 \\ 1 \\ 1 \\ 1 \end{pmatrix}.$$

On the other hand, the first two eigenvectors correspond to $\lambda = 2$, so $\boldsymbol{v}_4$ corresponds to $\lambda = -2$. Since the eigenspace associated to $\lambda = -2$ has dimension 2, the new eigenvector doesn't have to be orthogonal to the three given eigenvectors: we could have chosen any vector

$$a \begin{pmatrix} 0 \\ -1 \\ 1 \\ 0 \end{pmatrix} + b \begin{pmatrix} -1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = a\boldsymbol{v}_3 + b\boldsymbol{v}_4$$

with $b \neq 0$.

□

(Final May 16/17) Exercise 3(a)

A) Vector $\boldsymbol{y}$ is the element of S for $a = b = 0$, so it is a solution to the linear system.

B) On the one hand, it is clear that S is a plane, so the set of solutions to the homogeneous system has to be also a plane (dimension 2).

On the other hand, when two vectors $\boldsymbol{x}$ and $\boldsymbol{y}$ are solutions to the linear system, its difference $\boldsymbol{x} - \boldsymbol{y}$ is a solution to the associated homogeneous system since

$$\mathbf{A}(\boldsymbol{x} - \boldsymbol{y}) = \mathbf{A}\boldsymbol{x} - \mathbf{A}\boldsymbol{y} = \boldsymbol{b} - \boldsymbol{b} = \mathbf{0}.$$

Hence, is we susbtrac $\boldsymbol{y} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$, from each $\boldsymbol{x} \in S$ we get a solution to $\mathbf{A}\boldsymbol{x} = \mathbf{0}$. But it is clear that if we apply this operation to all vectors in S we get $\mathcal{N}$.

Hence we have:

1. All vectors in $\mathcal{N}$ are solutions to $\mathbf{A}\boldsymbol{x} = \mathbf{0}$
2. $\mathcal{N}$ is a subspace of dimension 2

therefore $\mathcal{N}$ must contain all solutions to the homogeneous system.

□

(Final May 16/17) Exercise 3(b)

We have already shown that

$$\mathcal{N} = \left\{ \boldsymbol{z} \;\middle|\; \boldsymbol{z} = \alpha \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + \beta \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}; \quad \forall \alpha, \beta \in \mathbb{R} \right\} = \left\{ \boldsymbol{x} \in \mathbb{R}^3 | \mathbf{A}\boldsymbol{x} = \mathbf{0} \right\}.$$

Hence, we just only need to find a basis for the orthogonal complement, and then multiply the parametric equations by the vectors of that basis. Let's do it using the Gaussian elimination:

$$\left[\begin{array}{ccc} 0 & 1 & 1 \\ 1 & 1 & 0 \\ \hline x & y & z \end{array}\right] \xrightarrow{\substack{\boldsymbol{\tau} \\ {[(-1)\mathbf{1}+\mathbf{2}]} \\ (-1)\mathbf{3}+\mathbf{2}}} \left[\begin{array}{ccc} 0 & 0 & 1 \\ 1 & 0 & 0 \\ \hline x & (-x + y - z) & z \end{array}\right]$$

hence

$$\begin{bmatrix} -1 & 1 & -1 \end{bmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \alpha 0 + \beta 0 \quad \Rightarrow \quad -x + y - z = 0$$

$\square$

(Final May 16/17) Exercise 3(c)

$$\mathbf{P} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 1 & 0 \end{bmatrix} \left(\begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 1 & 0 \end{bmatrix} \right)^{-1} \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 1 & 0 \end{bmatrix} \left(\frac{1}{3} \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \right) \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$
$$= \frac{1}{3} \begin{bmatrix} 2 & 1 & -1 \\ 1 & 2 & 1 \\ -1 & 1 & 2 \end{bmatrix}$$

$\square$

(Final May 16/17) Exercise 3(d)

It is easy to see that

$$\alpha \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ 4 \end{pmatrix}$$

has no solutions. If you can't see it try gaussian elimnation:

$$\left[\begin{array}{cc|c} 1 & 0 & -3 \\ 1 & 1 & -2 \\ 0 & 1 & -4 \\ \hline 1 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right] \xrightarrow{\underset{[(3)\mathbf{1}+\mathbf{3}]}{\boldsymbol{\tau}}} \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & -4 \\ \hline 1 & 0 & 3 \\ 0 & 1 & 0 \end{array}\right] \xrightarrow{\underset{[(-1)\mathbf{2}+\mathbf{3}]}{\boldsymbol{\tau}}} \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & -5 \\ \hline 1 & 0 & 3 \\ 0 & 1 & -1 \end{array}\right]$$

or compute the determinant of the augmented matrix

$$\begin{vmatrix} 1 & 0 & 3 \\ 1 & 1 & 2 \\ 0 & 1 & 4 \end{vmatrix} = 5.$$

Aunque lo mejor y más sencillo es ver que el vector $(3, 2, 4)$ sencilamente no verifica la ecuación cartesiana que define al conjunto $-x + y - z = 0$.

Hence, we have to find the orthogonal projection. Using the matrix projection we get:

$$\mathbf{P}\boldsymbol{d} = \frac{1}{3} \begin{bmatrix} 2 & 1 & -1 \\ 1 & 2 & 1 \\ -1 & 1 & 2 \end{bmatrix} \begin{pmatrix} 3 \\ 2 \\ 4 \end{pmatrix} = \frac{1}{3} \begin{pmatrix} 4 \\ 11 \\ 7 \end{pmatrix}.$$

$\square$

(Final May 16/17) Short questions set 1(a)

True: When columns of $\mathbf{Q}$ are eigenvector of $\mathbf{A}$, then $\mathbf{AQ} = \mathbf{QD}$, where $\mathbf{D}$ is a diagonal matrix with the eigenvalues on its main diagonal. If $\mathbf{Q}$ is orthogonal, then $\mathbf{QQ}^\intercal = \mathbf{I}$, so, multiplying by $\mathbf{Q}^\intercal$ we get $\mathbf{A} = \mathbf{QDQ}^\intercal$; therefore

$$\mathbf{A}^\intercal = \left(\mathbf{QDQ}^\intercal\right)^\intercal = \left(\mathbf{Q}^\intercal\right)^\intercal \mathbf{D}^\intercal \mathbf{Q}^\intercal = \mathbf{QDQ}^\intercal = \mathbf{A}.$$

$\square$

(Final May 16/17) Short questions set 1(b)

True: When a matrix is positive definite its determinant is positive, but for this matrix $\det \mathbf{A} < 0$. When a matrix is negative definite, the determinants of its principal minor with odd order are posite, but for this matrix $a_{11} < 0$.

$\square$

(Final May 16/17) Short questions set 1(c)

True: Both eigenspaces are orthogonal since

$$\begin{bmatrix} 2 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix};$$

and its dimmesions sum up 3.

$\square$

(Final May 16/17) Short questions set 1(d)

True: When $\mathbf{A}$ is full rank its eigenvalues are non null, and then eigenvalues of $\mathbf{A}^2$ are positive.

Another answer: $\mathbf{A} = \mathbf{A}^\mathsf{T}$ so $\mathbf{A}^2 = \mathbf{A}^\mathsf{T}\mathbf{A}$; then $\boldsymbol{x}\mathbf{A}^2\boldsymbol{x} = \boldsymbol{x}\mathbf{A}^\mathsf{T}\mathbf{A}\boldsymbol{x} = \big[\boldsymbol{y}\big]^\mathsf{T}\big[\boldsymbol{y}\big]$, where $\boldsymbol{y} = \mathbf{A}\boldsymbol{x}$. Since $\mathbf{A}$ is full rank, $\big[\boldsymbol{y}\big]^\mathsf{T}\big[\boldsymbol{y}\big] > 0$ for all $\boldsymbol{x} \neq \mathbf{0}$.

$\square$

(Final May 16/17) Short questions set 2(a)

Looking at the first row of $\mathbf{A}$ we deduce that

$$\det \mathbf{A} = \det \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 3 \\ 1 & 3 & 9 \end{bmatrix} = 3.$$

Of course, $\det \mathbf{A}^{-1} = \frac{1}{3}$.

$\square$

(Final May 16/17) Short questions set 2(b)

$$(\mathbf{A}^{-1})_{12} = \frac{\operatorname{cof}(\mathbf{A})_{21}}{\det \mathbf{A}} = \frac{-1}{3} \det \begin{bmatrix} 0 & 1 & 0 \\ 2 & 1 & 3 \\ 3 & 1 & 9 \end{bmatrix} = \frac{-1(-9)}{3} = 3.$$

$\square$

(Final May 16/17) Short questions set 3(a)

Since $\mathbf{MM} = \mathbf{M}$, and $\mathbf{M}$ is invertible, we can multiply both sides by $\mathbf{M}^{-1}$ in order to get

$$\mathbf{MM} = \mathbf{M} \quad \Rightarrow \quad \mathbf{MMM}^{-1} = \mathbf{MM}^{-1} \quad \Rightarrow \quad \mathbf{M} = \mathbf{I}.$$

$\square$

(Final May 16/17) Short questions set 3(b)

Since $P(\lambda) = \lambda^4 - 3\lambda^3 + 2\lambda^2 = \lambda^2(\lambda^2 - 3\lambda + 2)$; polynomial $P(\lambda)$ has roots 0, 0, 1 and 2.

Since eigenvalue 0 is repeated and since we don't known the dimension of the associated eigenspace, we don't known if $\mathbf{N}$ is diagonalizable.

$\square$

(Final May 16/17) Short questions set 3(c)

$\mathbf{B}^\mathsf{T} = \left(\mathbf{AA}^\mathsf{T}\right)^\mathsf{T} = \left(\mathbf{A}^\mathsf{T}\right)^\mathsf{T}\mathbf{A}^\mathsf{T} = \mathbf{AA}^\mathsf{T} = \mathbf{B}.$

$\square$

(Final May 16/17) Short questions set 3(d)

The four eigenvalues of $\mathbf{B}$ are 0, 2, 2 and 4. Son the eigenvalues of $(\mathbf{B} - 2\mathbf{I})$ are -2, 0, 0 and 2. Since two eigenvalues are non zero, the rank is two.

$\square$

(Final June 15/16) Exercise 1(a)

$$\left[\begin{array}{ccc|c} 2 & 1 & 1 & -2\alpha \\ 4 & 2 & 2 & -4\alpha \\ 6 & 2 & 3 & 0 \\ \hline 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \xrightarrow[\substack{(-1)\mathbf{1}+\mathbf{2} \\ (-2)\mathbf{1}+\mathbf{3}}]{\substack{\boldsymbol{\tau} \\ [\mathbf{1}\rightleftharpoons\mathbf{3}]}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & -2\alpha \\ 2 & 0 & 0 & -4\alpha \\ 3 & -1 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & -1 & -2 & 0 \end{array}\right] \xrightarrow[(-3)\mathbf{2}+\mathbf{1}]{\substack{\boldsymbol{\tau} \\ [(-1)\mathbf{2}]}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & -2\alpha \\ 2 & 0 & 0 & -4\alpha \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 \\ 3 & -1 & 0 & 0 \\ -2 & 1 & -2 & 0 \end{array}\right] \xrightarrow{\substack{\boldsymbol{\tau} \\ [(2\alpha)\mathbf{1}+\mathbf{4}]}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 \\ 3 & -1 & 0 & 6\alpha \\ -2 & 1 & -2 & -4\alpha \end{array}\right]$$

therefore, it is solvable for any α.

$\square$

(Final June 15/16) Exercise 1(b)

$$\boldsymbol{x} = \begin{pmatrix} 0 \\ 6\alpha \\ -4\alpha \end{pmatrix} + a \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix} \qquad \text{for all } a \in \mathbb{R}.$$

$\square$

(Final June 15/16) Exercise 1(c)

We have seen that rank is 2 (just only two pivots), so it is singular, and its determinant is zero.

□

(Final June 15/16) Exercise 2(a)

$$\left[\begin{array}{c}\mathbf{A}\\ \hline \mathbf{I}\end{array}\right] = \left[\begin{array}{ccc}1&1&0\\2&1&2\\2&0&3\\ \hline 1&0&0\\0&1&0\\0&0&1\end{array}\right] \xrightarrow[{[(-1)\mathbf{1}+\mathbf{2}]}]{\tau} \left[\begin{array}{ccc}1&0&0\\2&-1&2\\2&-2&3\\ \hline 1&-1&0\\0&1&0\\0&0&1\end{array}\right] \xrightarrow[{[(2)\mathbf{2}+\mathbf{3}]}]{\tau} \left[\begin{array}{ccc}1&0&0\\2&-1&0\\2&-2&-1\\ \hline 1&-1&-2\\0&1&2\\0&0&1\end{array}\right] = \left[\begin{array}{c}\mathbf{L}\\ \hline \mathbf{E}\end{array}\right]$$

So $\det(\mathbf{A}) = \det(\mathbf{L}) = 1 \cdot (-1) \cdot (-1) = 1$.

□

(Final June 15/16) Exercise 2(b)

$$\left[\begin{array}{c}\mathbf{A}\\ \hline \mathbf{I}\end{array}\right] \to \left[\begin{array}{c}\mathbf{L}\\ \hline \mathbf{E}\end{array}\right] = \left[\begin{array}{ccc}1&0&0\\2&-1&0\\2&-2&-1\\ \hline 1&-1&-2\\0&1&2\\0&0&1\end{array}\right] \xrightarrow[\substack{[(-1)\mathbf{3}]\\(2)\mathbf{3}+\mathbf{2}\\(-2)\mathbf{3}+\mathbf{1}}]{\tau} \left[\begin{array}{ccc}1&0&0\\2&-1&0\\0&0&1\\ \hline -3&3&2\\4&-3&-2\\2&-2&-1\end{array}\right] \xrightarrow[\substack{[(-1)\mathbf{2}]\\(-2)\mathbf{2}+\mathbf{1}}]{\tau} \left[\begin{array}{ccc}1&0&0\\0&1&0\\0&0&1\\ \hline 3&-3&2\\-2&3&-2\\-2&2&-1\end{array}\right] = \left[\begin{array}{c}\mathbf{I}\\ \hline \mathbf{A}^{-1}\end{array}\right]$$

So

$$\mathbf{A}^{-1} = \begin{bmatrix}3&-3&2\\-2&3&-2\\-2&2&-1\end{bmatrix}$$

□

(Final June 15/16) Exercise 2(c)

Since $\mathbf{A}$ is full row rank, the system is solvable for any $\boldsymbol{b} \in \mathbb{R}^3$.

Since $\mathbf{A}$ is full column rank, the system could have just only one solution. Never infinite.

□

(Final June 15/16) Exercise 2(d)

$$\mathbf{A}\boldsymbol{x} = \begin{bmatrix}1&1&0\\2&1&2\\2&0&3\end{bmatrix}\begin{pmatrix}1\\2\\-1\end{pmatrix} = \begin{pmatrix}3\\2\\-1\end{pmatrix} = \boldsymbol{b}.$$

□

(Final June 15/16) Exercise 3(a)

Yes, since it is a symmetric matrix.

□

(Final June 15/16) Exercise 3(b)

$\lambda = 1, 1, 1, 1, 6$.

Since $\mathbf{A} - \mathbf{I}$ has all equal columns, it has rank one. It follows that has the eigenvalue 1 with multiplicity four. The trace of $\mathbf{A}$ equals 10 so $10 - 4 = 6$ is the other eigenvalue.

□

(Final June 15/16) Exercise 3(c)

For $\lambda = 1$, since $\mathbf{A} - \mathbf{I}$ has all equal columns, we get:

$$\left\{\begin{pmatrix}-1\\1\\0\\0\\0\end{pmatrix};\ \begin{pmatrix}-1\\0\\1\\0\\0\end{pmatrix};\ \begin{pmatrix}-1\\0\\0\\1\\0\end{pmatrix};\ \begin{pmatrix}-1\\0\\0\\0\\1\end{pmatrix}\right\}$$

For the eigenvalue, $\lambda = 6$, we get

$$\mathsf{A} - 6\mathsf{I} = \begin{bmatrix} -4 & 1 & 1 & 1 & 1 \\ 1 & -4 & 1 & 1 & 1 \\ 1 & 1 & -4 & 1 & 1 \\ 1 & 1 & 1 & -4 & 1 \\ 1 & 1 & 1 & 1 & -4 \end{bmatrix};$$

and, since the sum of columns of $\mathsf{A} - 6\mathsf{I}$ is zero, a fifth linearly independent eigenvector is

$$\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}.$$

$\square$

(Final June 15/16) Short questions set 1(a)

Since we are searching a representation of a line in $\mathbb{R}^3$, a linear system with two equations is enough. On the one hand, rows of A must be orthogonal to the set of solutions of $\mathsf{A}\boldsymbol{x} = \boldsymbol{0}$ (a parallel line that goes through the origin); on the other hand, the line must go through $\boldsymbol{p}$ (so $\mathsf{A}\boldsymbol{p} = \boldsymbol{b}$). Hence, if we choose a coeficient matrix A whose rows are $\boldsymbol{u}$ and $\boldsymbol{v}$, an we compute the rigth hand side vector $\boldsymbol{b}$ multiplying A by $\boldsymbol{p}$;

$$\mathsf{A} = \begin{bmatrix} 7 & 3 & 0 \\ 4 & 0 & 3 \end{bmatrix}; \qquad \boldsymbol{b} = \mathsf{A}\boldsymbol{p} = \begin{bmatrix} 7 & 3 & 0 \\ 4 & 0 & 3 \end{bmatrix} \begin{pmatrix} 1 \\ -3 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ 7 \end{pmatrix};$$

then, an implicit (or cartesian) representation is $\begin{cases} 7x + 3y & = -2 \\ 4x \quad + 3z & = 4 \end{cases}.$

$\square$

(Final June 15/16) Short questions set 1(b)

We just need to find a perpendicular vector to $\boldsymbol{u}$ and $\boldsymbol{v}$ in $\mathbb{R}^3$

$$\left[\begin{array}{ccc} 7 & 3 & 0 \\ 4 & 0 & 3 \\ \hline 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right] \xrightarrow[{[(4)\mathbf{3}]}]{\substack{\boldsymbol{\tau} \\ [(3)\mathbf{1}] \\ \boldsymbol{\tau} \\ [(7)\mathbf{2}] \\ \boldsymbol{\tau}}} \left[\begin{array}{ccc} 21 & 21 & 0 \\ 12 & 0 & 12 \\ \hline 3 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 4 \end{array}\right] \xrightarrow{\substack{\boldsymbol{\tau} \\ [(-1)\mathbf{2}+\mathbf{1}] \\ (-1)\mathbf{3}+\mathbf{1}}} \left[\begin{array}{ccc} 0 & 21 & 0 \\ 0 & 0 & 12 \\ \hline 3 & 0 & 0 \\ -7 & 7 & 0 \\ -4 & 0 & 4 \end{array}\right]$$

Therefore, a parametric representation is

$$\boldsymbol{x} = \boldsymbol{p} + a \begin{pmatrix} 3 \\ -7 \\ -4 \end{pmatrix} \quad \Rightarrow \quad \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ -3 \\ 1 \end{pmatrix} + a \begin{pmatrix} 3 \\ -7 \\ -4 \end{pmatrix}.$$

$\square$

(Final June 15/16) Short questions set 2.

Consider the following linear combination: $\boldsymbol{0} = c_1\boldsymbol{u}_1 + c_2\boldsymbol{u}_2 + c_3\boldsymbol{u}_1$. Then, multiplying by $\boldsymbol{u}_1$ we conclude that

$$0 = \boldsymbol{u}_1\boldsymbol{0} = \boldsymbol{u}_1(c_1\boldsymbol{u}_1 + c_2\boldsymbol{u}_2 + c_3\boldsymbol{u}_1) = c_1\boldsymbol{u}_1\boldsymbol{u}_1 \Rightarrow c_1 = 0.$$

Do something similar for c_2 and c_3.

$\square$

(Final June 15/16) Short questions set 3(a)

False. For example $(1, 0, 0)$, $(-1, 0, 0)$ and $(0, 0, 0)$.

$\square$

(Final June 15/16) Short questions set 3(b)

True. If rank equals the number of columns, then all columns have a pivot after gaussian elimination. So in the search of a solution $\mathsf{A}\boldsymbol{x} = \boldsymbol{0}$, the only linear combination of pivot columns that equals $\boldsymbol{0}$ is the trivial one.

□

(Final June 15/16) Short questions set 3(c)

False. For example $\begin{bmatrix}1&1\\0&1\end{bmatrix}$ or $\begin{bmatrix}2&\\&1/2\end{bmatrix}$.

□

(Final June 15/16) Short questions set 3(d)

True. $1 = \det(\mathbf{I}) = \det(\mathbf{A}^\intercal\mathbf{A}) = \det(\mathbf{A}^\intercal)\cdot\det(\mathbf{A}) = \Big(\det(\mathbf{A})\Big)^2$, hence, the only posible values for $\det(\mathbf{A})$ are 1 or -1.

□

(Final June 15/16) Short questions set 3(e)

True. At most five of them $(\boldsymbol{\lambda}_5,\dots,\boldsymbol{\lambda}_n)$ are non-zero (since they are distinct). And hence, the corresponding eigenvectors $\boldsymbol{v}_5,\dots,\boldsymbol{v}_n$ are independent, and they are in $\mathcal{C}\left(\mathbf{A}\right)$ since $\mathbf{A}\frac{\boldsymbol{v}_i}{\lambda_i} = \boldsymbol{v}_i$. So that $\operatorname{rg}\left(\mathbf{A}\right) = \dim\mathcal{C}\left(\mathbf{A}\right) \geq 5$.

□

(Final June 15/16) Short questions set 4(a)

We need a rank 3 matrix; by Gaussian elimination we get:

$$\left[\begin{array}{c}\mathbf{A}\\\hline\mathbf{I}\end{array}\right] = \left[\begin{array}{cccc}1&0&1&1\\1&1&2&3\\a&1&1&2\\\hline 1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{array}\right] \xrightarrow[(-1)\mathbf{4}+\mathbf{1}]{\substack{\boldsymbol{\tau}\\ [(-1)\mathbf{4}+\mathbf{3}]}} \left[\begin{array}{cccc}0&0&0&1\\-2&1&-1&3\\a-2&1&-1&2\\\hline 1&0&0&0\\0&1&0&0\\0&0&1&0\\-1&0&-1&1\end{array}\right] \xrightarrow[(-2)\mathbf{3}+\mathbf{1}]{\substack{\boldsymbol{\tau}\\ [(1)\mathbf{3}+\mathbf{2}]}} \left[\begin{array}{cccc}0&0&0&1\\0&0&-1&3\\a&0&-1&2\\\hline 1&0&0&0\\0&1&0&0\\-2&1&1&0\\1&-1&-1&1\end{array}\right] = \left[\begin{array}{c}\mathbf{L}\\\hline\mathbf{E}\end{array}\right]$$

therefore, if $a \neq 0$ the rank of $\mathbf{A}$ is 3, and the dimension of $\mathcal{N}\left(\mathbf{A}\right)$ is one.

□

(Final June 15/16) Short questions set 4(b)

When $a = 0$; in that case $\dim\mathcal{N}\left(\mathbf{A}\right) = 2$.

□

(Final May 15/16) Exercise 1(a)

A parametric representation of the line is

$$\begin{pmatrix}x\\y\\z\end{pmatrix} = \boldsymbol{a} + \alpha(\boldsymbol{a}-\boldsymbol{b}) = \begin{pmatrix}1\\0\\3\end{pmatrix} + \alpha\begin{pmatrix}4/3\\0\\4\end{pmatrix}.$$

□

(Final May 15/16) Exercise 1(b)

Since $(-3,0,1)$ and $(0,1,0)$ are orthogonal to $(\frac{4}{3},0,4)$, a cartesian representation of the line is

$$\begin{bmatrix}-3&0&1\\0&1&0\end{bmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{bmatrix}-3&0&1\\0&1&0\end{bmatrix}\begin{pmatrix}1\\0\\3\end{pmatrix} + \alpha\begin{bmatrix}-3&0&1\\0&1&0\end{bmatrix}\begin{pmatrix}4/3\\0\\4\end{pmatrix} \quad\Rightarrow\quad \begin{cases}-3x & +z=0\\ \quad y & \quad =0\end{cases}.$$

□

(Final May 15/16) Exercise 1(c)

Yes. It is the set of solutions of the homogeneous system

$$\begin{cases}-3x & +z=0\\ \quad y & \quad =0\end{cases};$$

i.e., it is a line through the origin.

□

(Final May 15/16) Exercise 1(d)

The line is the spam of $\begin{pmatrix}4/3\\0\\4\end{pmatrix}$ or dividing by 4 and multiplying by 3, it is the line spam by $\begin{pmatrix}1\\0\\3\end{pmatrix}$; then

$$\mathbf{P} = \begin{bmatrix}1\\0\\3\end{bmatrix}\left(\begin{bmatrix}1 & 0 & 3\end{bmatrix}\begin{bmatrix}1\\0\\3\end{bmatrix}\right)^{-1}\begin{bmatrix}1 & 0 & 3\end{bmatrix} = \frac{1}{10}\begin{bmatrix}1\\0\\3\end{bmatrix}\begin{bmatrix}1 & 0 & 3\end{bmatrix} = \frac{1}{10}\begin{bmatrix}1 & 0 & 3\\0 & 0 & 0\\3 & 0 & 9\end{bmatrix}.$$

□

(Final May 15/16) Exercise 1(e)

It is the proyection of $\boldsymbol{z}$ on the line spaned by $(1, 0, 3)$:

$$\boldsymbol{p} = \mathbf{P}\boldsymbol{z} = \frac{1}{10}\begin{bmatrix}1 & 0 & 3\\0 & 0 & 0\\3 & 0 & 9\end{bmatrix}\begin{pmatrix}2\\2\\2\end{pmatrix} = \frac{1}{10}\begin{pmatrix}8\\0\\24\end{pmatrix}.$$

□

(Final May 15/16) Exercise 2(a)

$|A| = 2\begin{vmatrix}1 & 1\\1 & 2\end{vmatrix} - (1-m)\begin{vmatrix}1-m & 0\\1 & 2\end{vmatrix} = 2 - 2(1-m)^2 = (4-2m)m$. $\mathbf{A}$ is singular when $m = 0$ or $m = 2$.

□

(Final May 15/16) Exercise 2(b)

Since $\mathbf{A} = \begin{bmatrix}2 & 1-m & 0\\1-m & 1 & 1\\0 & 1 & 2\end{bmatrix}$, for $\mathbf{B} = \begin{bmatrix}1-m & 2 & 0\\1 & 1-m & 1\\1 & 0 & 2\end{bmatrix}$ and $\mathbf{C} = \begin{bmatrix}2 & 1-m & 0\\1-m & 1 & 1/2\\0 & 1 & 1\end{bmatrix}$ their determinants are $|\mathbf{B}| = -|\mathbf{A}|$ and $|\mathbf{C}| = \frac{1}{2}|\mathbf{A}|$.

□

(Final May 15/16) Exercise 2(c)

For $m = 1$, determinant of $\mathbf{A}$ is 2. Using the Cramer rule we get

$$x_1 = \frac{\begin{vmatrix}3 & 0 & 0\\4 & 1 & 1\\2 & 1 & 2\end{vmatrix}}{\det(\mathbf{A})} = \frac{3}{2}; \qquad x_2 = \frac{\begin{vmatrix}2 & 3 & 0\\0 & 4 & 1\\0 & 2 & 2\end{vmatrix}}{\det(\mathbf{A})} = \frac{2(6)}{2} = 6; \qquad x_3 = \frac{\begin{vmatrix}2 & 0 & 3\\0 & 1 & 4\\0 & 1 & 2\end{vmatrix}}{\det(\mathbf{A})} = \frac{2(-2)}{2} = -2.$$

□

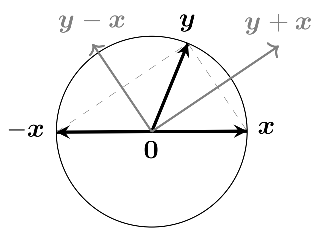
(Final May 15/16) Exercise 3(a)

Since $\|\boldsymbol{x}\| = \|\boldsymbol{y}\|$ means $\boldsymbol{x}\cdot\boldsymbol{x} = \boldsymbol{y}\cdot\boldsymbol{y}$; we get

$$\begin{aligned}(\boldsymbol{y}+\boldsymbol{x})\cdot(\boldsymbol{y}-\boldsymbol{x}) &= \boldsymbol{y}\cdot\boldsymbol{y} - \boldsymbol{y}\cdot\boldsymbol{x} + \boldsymbol{x}\cdot\boldsymbol{y} - \boldsymbol{x}\cdot\boldsymbol{x} \\ &= \boldsymbol{y}\cdot\boldsymbol{y} - \boldsymbol{x}\cdot\boldsymbol{x} && \text{since } \boldsymbol{y}\cdot\boldsymbol{x} = \boldsymbol{x}\cdot\boldsymbol{y} \\ &= 0 && \text{since } \|\boldsymbol{x}\| = \|\boldsymbol{y}\|.\end{aligned}$$

□

(Final May 15/16) Exercise 3(b)



□

(Final May 15/16) Exercise 3(c)

On the one hand, segment $[\boldsymbol{ab}]$ is parallel to $\boldsymbol{y}+\boldsymbol{x}$, and segment $[\boldsymbol{bc}]$ is parallel to $\boldsymbol{y}-\boldsymbol{x}$; on the other hand, both segments have length equal to the radius of the circle (so $\|\boldsymbol{x}\| = \|\boldsymbol{y}\|$); therefore, by part (a), they must be perpendicular.

$\square$

(Final May 15/16) Short questions set 1(a)

Since $\mathbf{C}$ is symmetric:

$$\mathbf{M}^\mathsf{T} = (\mathbf{A}^\mathsf{T}\mathbf{C}\mathbf{A})^\mathsf{T} = \mathbf{A}^\mathsf{T}\mathbf{C}^\mathsf{T}\big(\mathbf{A}^\mathsf{T}\big)^\mathsf{T} = \mathbf{A}^\mathsf{T}\mathbf{C}\mathbf{A} = \mathbf{M}.$$

Hence, $\mathbf{M}$ is symmetric.

$\square$

(Final May 15/16) Short questions set 1(b)

Since $\mathbf{C}$ is symmetric and positive defined, then $\mathbf{C} = \mathbf{Q}\mathbf{D}\mathbf{Q}^\mathsf{T}$; where $\mathbf{D}$ is a diagonal matrix and all entries in the main diagonal are greater than zero. So

$$\boldsymbol{x}\mathbf{M}\boldsymbol{x} = \boldsymbol{x}\mathbf{A}^\mathsf{T}\mathbf{C}\mathbf{A}\boldsymbol{x} = \boldsymbol{x}\mathbf{A}^\mathsf{T}\mathbf{Q}\mathbf{D}\mathbf{Q}^\mathsf{T}\mathbf{A}\boldsymbol{x}.$$

If we denote $\mathbf{B} = \mathbf{Q}^\mathsf{T}\mathbf{A}$ we get

$$\boldsymbol{x}\mathbf{M}\boldsymbol{x} = \boldsymbol{x}\mathbf{B}^\mathsf{T}\mathbf{D}\mathbf{B}\boldsymbol{x} = (\mathbf{B}\boldsymbol{x})^\mathsf{T}\mathbf{D}(\mathbf{B}\boldsymbol{x}) \geq 0,$$

since it is a sum of squares. Quadratic form $\boldsymbol{x}\mathbf{M}\boldsymbol{x}$ will be defined (i.e., $\boldsymbol{x}\mathbf{M}\boldsymbol{x} = 0$ if and only if $\boldsymbol{x} = \mathbf{0}$) if $\mathbf{B}\boldsymbol{x} \neq \mathbf{0}$ for all $\boldsymbol{x} \neq \mathbf{0}$, that is, when $\mathbf{B}$ is a full rank matrix.

$\square$

(Final May 15/16) Short questions set 1(c)

If $\mathbf{M}$ is not positive definite, then it is positive semi-definite, that is, at least one eigenvalue is zero, and the other are posive or equal to zero. Therefore, the answer is $\lambda = 0$.

$\square$

(Final May 15/16) Short questions set 2(a)

Eigenvalues: $\lambda = 2$ and $\lambda = 4$. For $\lambda = 2$ it is easy to check that $\dim \mathcal{N}(\mathbf{A} - 2\mathbf{I}) = 1$ (only one free column), but there are two eigenvalues equal to 2. Hence, $\mathbf{A}$ is not diagonalizable.

$\square$

(Final May 15/16) Short questions set 2(b)

It is invertible since $|\mathbf{A}| = 16 \neq 0$.

$$\left[\begin{array}{c}\mathbf{A}\\ \hline \mathbf{I}\end{array}\right] = \left[\begin{array}{ccc}2&4&2\\0&4&2\\0&0&2\\ \hline 1&0&0\\0&1&0\\0&0&1\end{array}\right] \xrightarrow[(-1)\mathbf{1}+\mathbf{3}]{\tau_{[(-2)\mathbf{1}+\mathbf{2}]}} \left[\begin{array}{ccc}2&0&0\\0&4&2\\0&0&2\\ \hline 1&-2&-1\\0&1&0\\0&0&1\end{array}\right] \xrightarrow{\tau_{[(-1/2)\mathbf{2}+\mathbf{3}]}} \left[\begin{array}{ccc}2&0&0\\0&4&0\\0&0&2\\ \hline 1&-2&0\\0&1&-1/2\\0&0&1\end{array}\right] \xrightarrow[\substack{(1/4)\mathbf{2}\\(1/2)\mathbf{3}}]{\tau_{[(1/2)\mathbf{1}]}} \left[\begin{array}{ccc}1&0&0\\0&1&0\\0&0&1\\ \hline 1/2&-1/2&0\\0&1/4&-1/4\\0&0&1/2\end{array}\right] = \left[\begin{array}{c}\mathbf{I}\\ \hline \mathbf{A}^{-1}\end{array}\right]$$

$\square$

(Final May 15/16) Short questions set 3(a)

True. If $\boldsymbol{v}_1$, $\boldsymbol{v}_2$,…,$\boldsymbol{v}_k$ are linearly independent, then, they are a basis, so $\dim(\mathcal{V}) = k$. Otherwise, we can span $\mathcal{V}$ with less than k vectors, and therefore $\dim(\mathcal{V}) < k$.

$\square$

(Final May 15/16) Short questions set 3(b)

True. If $\boldsymbol{v}_1$, $\boldsymbol{v}_2$,…,$\boldsymbol{v}_k$ span $\mathcal{V}$, then, they are a basis, so $\dim(\mathcal{V}) = k$. Otherwise, there are vectors in $\mathcal{V}$ that are not a linear combination of $\boldsymbol{v}_1$, $\boldsymbol{v}_2$,…,$\boldsymbol{v}_k$ and therefore $\dim(\mathcal{V}) > k$.

$\square$

(Final May 15/16) Short questions set 3(c)

True. There is at least one free column, so, if the system is solvable, the solution set has infinite vectors.

$\square$

(Final May 15/16) Short questions set 3(d)

False. For example

$$\begin{bmatrix}1&1&1\\1&1&1\\1&1&1\\1&1&1\end{bmatrix}\boldsymbol{x} = \begin{pmatrix}1\\1\\1\\1\end{pmatrix}$$

has infinite solutions.

□

(Final May 15/16) Short questions set 3(e)

False. The scalar product (dot product) is an scalar (a real number).

□

(Final June 14/15) Exercise 1(a)

The system could have no solution; for example $\begin{bmatrix} 1 & 1 & 0 & 0 \\ 2 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ Note that rank of $\mathbf{A}$ is one, but the rank of the augmented matrix is two.

□

(Final June 14/15) Exercise 1(b)

Since there are more equations than unknowns, if the system has a solution, that solution is not unique.

□

(Final June 14/15) Exercise 1(c)

The rigth hand side vector $\boldsymbol{b}$ must be a linear combination of the columns of $\mathbf{A}$; in other words, the matrices $\mathbf{A}$ and $[\mathbf{A}|\boldsymbol{b}]$ must have the same rank.

□

(Final June 14/15) Exercise 1(d)

Since $\boldsymbol{b}$ belongs to $\mathbb{R}^3$, the rank of $\mathbf{A}$ must be 3.

□

(Final June 14/15) Exercise 1(e)

The answer is YES. If some columns of $\mathbf{A}^\intercal$ are linearly dependent, and $\boldsymbol{c}$ is a linear combination of those dependent columns, then the system has an infinite number of solutions. For example:

$$\text{If } \mathbf{A} = \begin{bmatrix} 1 & 2 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 1 & 2 & 3 & 4 \end{bmatrix} \quad \text{and} \quad \boldsymbol{c} = \begin{pmatrix} 2 \\ 4 \\ 0 \\ 0 \end{pmatrix},$$

then the linear system

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 0 & 0 & 3 \\ 0 & 0 & 4 \end{bmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \\ 0 \\ 0 \end{pmatrix}$$

has multiple solutions, for example $\boldsymbol{y} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$, or $\begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}$, or $\begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}$, etc...

□

(Final June 14/15) Exercise 2(a)

$\mathbf{A} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \text{sum of the columns} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$

□

(Final June 14/15) Exercise 2(b)

$\mathbf{A}^2 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \mathbf{A} \cdot \mathbf{A} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \mathbf{A} \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$

□

(Final June 14/15) Exercise 2(c)

The dimension is at least 1 (because $\mathbf{A}$ is square and we know that $(1;1;1)$ is solution to $\mathbf{A}\boldsymbol{x} = \mathbf{0}$).

□

(Final June 14/15) Exercise 2(d)

$\mathbf{A}$ is singular, and therefore $\lambda = 0$ is an eigenvalue of $\mathbf{A}$. Hence, $\lambda^3 = 0$ is an eigenvalue of $\mathbf{A}^3$.

□

(Final June 14/15) Exercise 3(a)

$\det(\mathbf{A}-\lambda\mathbf{I}) = \begin{vmatrix} 3-\lambda & 4 & 6 \\ 0 & 1-\lambda & 0 \\ -1 & -2 & -2-\lambda \end{vmatrix} = (1-\lambda)(1-\lambda)\lambda$, so the eigenvalues of $\mathbf{A}$ are 1, 1, and 0.

□

(Final June 14/15) Exercise 3(b)

On the one hand,

$$\mathbf{A}\boldsymbol{x} = \begin{bmatrix} 3 & 4 & 6 \\ 0 & 1 & 0 \\ -1 & -2 & -2 \end{bmatrix} \boldsymbol{x} = \mathbf{0} \qquad \text{has special solution } \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}.$$

On the other hand,

$$\mathbf{A}\boldsymbol{x} = \begin{bmatrix} 2 & 4 & 6 \\ 0 & 0 & 0 \\ -1 & -2 & -3 \end{bmatrix} \boldsymbol{x} = \mathbf{0} \qquad \text{has special solutions } \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} \text{ and } \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix}.$$

So one such basis is

$$\boldsymbol{v}_1 = \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix};\ \boldsymbol{v}_2 = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix};\ \boldsymbol{v}_3 = \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix}.$$

□

(Final June 14/15) Exercise 3(c)

First method. We solve $\mathbf{S}\boldsymbol{v} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$:

$$\left[\begin{array}{c|c} \mathbf{S} & \text{-}\boldsymbol{b} \\ \hline \mathbf{I} & \mathbf{0} \end{array}\right] = \left[\begin{array}{ccc|c} -2 & -2 & -3 & -1 \\ 0 & 1 & 0 & -1 \\ 1 & 0 & 1 & -1 \\ \hline 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \xrightarrow{\substack{\boldsymbol{\tau} \\ [(-1)\mathbf{3}+\mathbf{1}] \\ (-1)\mathbf{3}+\mathbf{2}}} \left[\begin{array}{ccc|c} 1 & 1 & -3 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & -1 & 1 & -1 \\ \hline 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & -1 & 1 & 0 \end{array}\right] \xrightarrow{\substack{\boldsymbol{\tau} \\ [(-1)\mathbf{1}+\mathbf{2}] \\ (3)\mathbf{1}+\mathbf{3} \\ (1)\mathbf{1}+\mathbf{4}}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & -1 & 1 & -1 \\ \hline 1 & -1 & 3 & 1 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & -2 & -1 \end{array}\right]$$

$$\xrightarrow{\substack{\boldsymbol{\tau} \\ [(1)\mathbf{2}+\mathbf{4}]}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & -2 \\ \hline 1 & -1 & 3 & 0 \\ 0 & 1 & 0 & 1 \\ -1 & 0 & -2 & -1 \end{array}\right] \xrightarrow{\substack{\boldsymbol{\tau} \\ [(2)\mathbf{3}+\mathbf{4}] \\ (1)\mathbf{3}+\mathbf{2}}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \hline 1 & 2 & 3 & 6 \\ 0 & 1 & 0 & 1 \\ -1 & -2 & -2 & -5 \end{array}\right] = \left[\begin{array}{c|c} \mathbf{I} & \mathbf{0} \\ \hline \mathbf{S}^{\text{-}1} & \boldsymbol{x}_p \end{array}\right]$$

Therefore, $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = 6\boldsymbol{v}_1 + \boldsymbol{v}_2 - 5\boldsymbol{v}_3 = 6\begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} - 5\begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix}$. So, $\mathbf{A}^{99}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \mathbf{A}^{99}\left(6\boldsymbol{v}_1 + \boldsymbol{v}_2 - 5\boldsymbol{v}_3\right)$, and therefore,

$$\mathbf{A}^{99}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \mathbf{A}^{99}(6\boldsymbol{v}_1) + \mathbf{A}^{99}(\boldsymbol{v}_2) + \mathbf{A}^{99}(-5\boldsymbol{v}_3) = 0^{99}(6\boldsymbol{v}_1) + 1^{99}(\boldsymbol{v}_2) + 1^{99}(-5\boldsymbol{v}_3) = \mathbf{0} + \boldsymbol{v}_2 - 5\boldsymbol{v}_3 = \begin{pmatrix} 13 \\ 1 \\ -5 \end{pmatrix}.$$

Second method. In this case the factorization $\mathbf{A} = \mathbf{S}\mathbf{D}\mathbf{S}^{\text{-}1}$, is: $\mathbf{A} = \mathbf{S}\begin{bmatrix} 0 & & \\ & 1 & \\ & & 1 \end{bmatrix}\mathbf{S}^{\text{-}1}$.

$$\mathbf{A}^{99} = \mathbf{S}\begin{bmatrix} 0 & & \\ & 1 & \\ & & 1 \end{bmatrix}^{99}\mathbf{S}^{\text{-}1} = \mathbf{S}\begin{bmatrix} 0^{99} & & \\ & 1^{99} & \\ & & 1^{99} \end{bmatrix}\mathbf{S}^{\text{-}1} = \mathbf{S}\begin{bmatrix} 0 & & \\ & 1 & \\ & & 1 \end{bmatrix}\mathbf{S}^{\text{-}1} = \mathbf{S}\mathbf{D}\mathbf{S}^{\text{-}1} = \mathbf{A}.$$

So

$$\mathbf{A}^{99}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \mathbf{A}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 13 \\ 1 \\ -5 \end{pmatrix}.$$

$\square$

(Final June 14/15) Short questions set 1(a)

There are not enough independent eigenvectors to form an invertible matrix $\mathbf{S}$ with eigenvectors as its columns.

$\square$

(Final June 14/15) Short questions set 1(b)

$\lambda_1 = \lambda_2 = \sqrt{2}$. The set of all the eigenvectors is: $\mathcal{N}(\mathbf{A} - \sqrt{2}\mathbf{I}) = \mathcal{N}\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right) = \text{span}\left\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right\}$.

$\square$

(Final June 14/15) Short questions set 1(c)

$\begin{vmatrix} 2 \end{vmatrix} > 0;\quad \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} > 0;\quad \begin{vmatrix} 2 & 1 & b \\ 1 & 2 & 1 \\ b & 1 & 2 \end{vmatrix} = -2b^2 + 2b + 4$; parabolic function that cross the x axis in -1 and 2.

- If $-1 < b < 2$ positive defined
- If $b = -1$ or $b = 2$ positive semi-defined
- Not-definite in other cases

$\square$

(Final June 14/15) Short questions set 2(a)

$\det \mathbf{A} \;=\; 2 + c^2 + 2c^2 - 4 - c^2 - c^2 \;=\; -2 + c^2$, so $\det \mathbf{A} = 0$ for $c = \pm\sqrt{2}$.

$\square$

(Final June 14/15) Short questions set 2(b)

$$\left[\begin{array}{c} \mathbf{A} \\ \hline \mathbf{I} \end{array}\right] = \left[\begin{array}{ccc} 1 & 0 & 2 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \\ \hline 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right] \xrightarrow[{[(-2)\mathbf{1}+\mathbf{3}]}]{\mathcal{T}} \left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 1 & 0 & -1 \\ \hline 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right] \xrightarrow[{[(-1)\mathbf{3}]}]{\mathcal{T}} \left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \\ \hline 1 & 0 & 2 \\ 0 & 1 & -0 \\ 0 & 0 & -1 \end{array}\right] \xrightarrow[{(1/2)\mathbf{2}}]{\substack{\mathcal{T} \\ [(-1)\mathbf{3}+\mathbf{1}]}} \left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ \hline -1 & 0 & 2 \\ 0 & 1/2 & 0 \\ 1 & 0 & -1 \end{array}\right] = \left[\begin{array}{c} \mathbf{I} \\ \hline \mathbf{A}^{-1} \end{array}\right].$$

$\square$

(Final June 14/15) Short questions set 2(c)

From part a), we already know $\mathbf{A}$ is full rank when $c = 1$. Hence the system has only one solution:

$$\left[\begin{array}{c|c} \mathbf{A} & \boldsymbol{-b} \\ \hline \mathbf{I} & \mathbf{0} \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 1 & 2 & -4 \\ 1 & 2 & 1 & -1 \\ 1 & 1 & 1 & -2 \\ \hline 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \xrightarrow[{[(2)\mathbf{3}+\mathbf{4}]}]{\mathcal{T}} \left[\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 1 & 2 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ \hline 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \end{array}\right] \xrightarrow[{[(-1)\mathbf{1}+\mathbf{2}]}]{\mathcal{T}} \left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ \hline 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \end{array}\right] \xrightarrow[{[(-1)\mathbf{2}+\mathbf{4}]}]{\mathcal{T}} \left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ \hline 1 & -1 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{array}\right] = \left[\begin{array}{c|c} \mathbf{B} & \mathbf{0} \\ \hline \mathbf{E} & \boldsymbol{x_p} \end{array}\right]$$

Hence, the solution is $x_1 = 1$, $x_2 = -1$, $x_3 = 2$.

$\square$

(Final June 14/15) Short questions set 3(a)

True. $(\mathbf{A}^2)^{\intercal} = (\mathbf{A}\mathbf{A})^{\intercal} = \mathbf{A}^{\intercal}\mathbf{A}^{\intercal} = \mathbf{A}\mathbf{A} = \mathbf{A}^2$.

$\square$

(Final June 14/15) Short questions set 3(b)

False. $\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ is not symmetric, but $\mathbf{A}^2 = \mathbf{0}$.

$\square$

(Final June 14/15) Short questions set 3(c)

False. The squared matrix is singular, so there is a free column.

$\square$

(Final June 14/15) Short questions set 3(d)

False. Example: If $\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$; then $\mathbf{A}^{-1} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$. The eigenvalies of $\mathbf{A}$ are $\lambda_1 = \lambda_2 = 1$, but its corresponding eigenspace is only the line spanned by $\boldsymbol{v} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

□

(Final May 14/15) Exercise 1(a)

$\mathcal{C}\left(\mathbf{A}\right)$ is a subspace of $\mathbb{R}^5$ but, since the system has a solution for every vector $\boldsymbol{b}$ in $\mathbb{R}^5$, each vector $\boldsymbol{b} \in \mathbb{R}^5$ is also in the column space of $\mathbf{A}$. Hence $\mathcal{C}\left(\mathbf{A}\right) = \mathbb{R}^5$ and therefore the columns of $\mathbf{A}$ are a generating system of $\mathbb{R}^5$. So $dim(\mathcal{C}\left(\mathbf{A}\right)) = \operatorname{rg}\left(\mathbf{A}\right) = 5$.

□

(Final May 14/15) Exercise 1(b)

Since the rank is 5, rows must be linearly independent.

□

(Final May 14/15) Exercise 1(c)

Since there are 7 columns, and the rank of $\mathbf{A}$ is 5, the nullspace must have dimension $7 - 5 = 2$.

□

(Final May 14/15) Exercise 1(d)

Since that $\mathbf{A}$ and $\mathbf{A}^\intercal$ have rank 5, the zero vector $\mathbf{0}$ is the only solution to $\mathbf{A}^\intercal \boldsymbol{x} = \boldsymbol{b}$.

□

(Final May 14/15) Exercise 1(e)

False. There are 7 columns, but the rank is only 5. Therefore, those columns are linearly dependent.

□

(Final May 14/15) Exercise 2(a)

On the one hand, $\mathbf{A}\boldsymbol{x}_3$ is equal to the third column of $\mathbf{A}$, meaning $\mathbf{A}\boldsymbol{x}_3 = \mathbf{A}_{|3}$.
On the other hand,$\boldsymbol{x}_3$ is the eigenvector associated to the eigenvalue 0, meaning $\mathbf{A}\boldsymbol{x}_3 = 0\boldsymbol{x}_3 = \mathbf{0}$.
Hence, the third column of $\mathbf{A}$ is the zero vector: $\mathbf{A}_{|3} = \mathbf{0}$.

□

(Final May 14/15) Exercise 2(b)

Since $\mathbf{A}$ has three not repeated eigenvalues, it must be a 3 by 3 matrix. Since we already know three linearly independent eigenvectors, the matrix $\mathbf{A}$ is diagonalizable; so $\mathbf{A} = \mathbf{SDS}^{-1}$ where:

$$\mathbf{D} = \begin{bmatrix} 3 & & \\ & 1 & \\ & & 0 \end{bmatrix}; \quad \mathbf{S} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}.$$

We first need to find $\mathbf{S}^{-1}$:

$$\begin{bmatrix} \mathbf{S} \\ \mathbf{I} \end{bmatrix} = \left[\begin{array}{ccc} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \\ \hline 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right] \xrightarrow[(-1)\mathbf{3}+\mathbf{2}]{\underset{[(-1)\mathbf{3}+\mathbf{1}]}{\tau}} \left[\begin{array}{ccc} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \\ \hline 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & -1 & 1 \end{array}\right] \xrightarrow{\underset{[(-1)\mathbf{2}+\mathbf{1}]}{\tau}} \left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ \hline 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{array}\right] = \left[\begin{array}{c} \mathbf{I} \\ \hline \mathbf{S}^{-1} \end{array}\right].$$

So

$$\mathbf{A} = \mathbf{SDS}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 3 & & \\ & 1 & \\ & & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 0 \\ 2 & 1 & 0 \\ 2 & 1 & 0 \end{bmatrix}$$

.

□

(Final May 14/15) Exercise 2(c)

$$\mathbf{D}^\intercal = (\mathbf{S}^{-1}\mathbf{AS})^\intercal = (\mathbf{S}^\intercal \mathbf{A}^\intercal \left(\mathbf{S}^\intercal\right)^{-1}) = \mathbf{D}$$

Hence,

$$\mathbf{A}^\intercal = \left(\mathbf{S}^\intercal\right)^{-1} \mathbf{DS}^\intercal$$

it follows that columns of $\left(\mathbf{S}^\intercal\right)^{-1} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$ are eigenvectors of $\mathbf{A}^\intercal$:

$$\boldsymbol{y}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}; \quad \boldsymbol{y}_2 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}; \quad \boldsymbol{y}_3 = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}.$$

□

(Final May 14/15) Exercise 3(a)

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & -0 \\ 2 & 1 & 2 & -a \\ 1 & a & c & -0 \\ \hline 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \xrightarrow[(-1)\mathbf{1}+\mathbf{3}]{\overset{\tau}{[(-2)\mathbf{1}+\mathbf{2}]}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 2 & -3 & 0 & -a \\ 1 & a-2 & c-1 & 0 \\ \hline 1 & -2 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \xrightarrow[(a)\mathbf{2}+\mathbf{4}]{\overset{\tau}{[(-1/3)\mathbf{2}]}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 1 & (2-a)/3 & c-1 & a(2-a)/3 \\ \hline 1 & 2/3 & -1 & 2a/3 \\ 0 & -1/3 & 0 & -a/3 \\ 0 & 0 & 1 & 0 \end{array}\right].$$

On the one hand, in order to have more than one solution, the rank of $\mathbf{A}$ should be 2; meaning that c should be equal to one ($c = 1$).

On the other hand, if $a = 0$ the system is homogeneous (so it is solvable): (for example the third column minus the first one); But there is also a solution when $a = 2$, (for example 4/3 of the first column minus 2/3 of the second one).

Hence, $c = 1$; and $a = 0$ or 2.

□

(Final May 14/15) Exercise 3(b)

We need a rank 2 coeficient matrix $\mathbf{A}$, but rank 3 augmented matrix; so $c = 1$, and $a \notin \{0, 2\}$.

□

(Final May 14/15) Exercise 3(c)

In this case matrix $\mathbf{A}$ should be rank 1. But the coeficient matrix has two pivots. So it is not possible.

□

(Final May 14/15) Exercise 3(d)

Here the coeficient matrix $\mathbf{A}$ should be rank 3. So $c \neq 1$.

□

(Final May 14/15) Exercise 3(e)

The second column of $\mathbf{A}$ is not a linear combination of the other columns, since the second column will always have a pivot.

□

(Final May 14/15) Short questions set 1(a)

$\det(\mathbf{A}) = 5x^2 - 6x + 0 - 9x + 10x - 0 = 5x^2 - 5x = 5x(x-1) = 0$. Therefore, $x = 0, x = 1$.

□

(Final May 14/15) Short questions set 1(b)

The main sub-determinants must be positive, therefore

- $|x| > 0 \Rightarrow; x > 0$.
- $\begin{vmatrix} x & 1 \\ 1 & x \end{vmatrix} > 0 \Rightarrow x^2 - 1 > 0 \Rightarrow |x| > 1$. Since x must be positive, then $x > 1$.
- $\det(\mathbf{B}) = x^2 - 2x + 1 = (x-1)(x-1) > 0$; so $x > 1$.

From all those three conditions the matrix is positive definite if and only if $x > 1$.

□

(Final May 14/15) Short questions set 1(c)

The first and third sub determinants should be negative, and the second one should be positive

- $|x| < 0 \Rightarrow; x < 0$.
- $\begin{vmatrix} x & 1 \\ 1 & x \end{vmatrix} > 0 \Rightarrow x^2 - 1 > 0 \Rightarrow |x| > 1$. Since $x < 0$, then $x < -1$.

- $\det(\mathbf{B}) = x^2 - 2x + 1 = (x-1)(x-1) = (x-1)^2$; But this cannot be negative. It follows that the matrix cannot be negative definite.

$\square$

(Final May 14/15) Short questions set 2(a)

True. Matrix $\mathbf{A}$ has 7 distinct eigenvalues $\lambda = 0, 1, -1, \sqrt{2}, -\sqrt{2}, \sqrt{3}, -\sqrt{3}$.

$\square$

(Final May 14/15) Short questions set 2(b)

True. Suppose -3 is the eigenvalue with a 3-dimensional eigenspace. Then it must occur with multiplicity (at least) 3 in the characteristic polynomial $p(\cdot)$ so $p(\lambda)$ contains the factors $(\lambda+3)^3(\lambda-2)(\lambda-7)$ and, since we know the degree of $p(\cdot)$ must be 5, there can be no other roots. In particular, 0 cannot be a root so 0 is not an eigenvalue and $\mathbf{A}$ must be invertible. [Obviously, the same idea works whichever of the eigenvalues has a 3-dimensional eigenspace].

$\square$

(Final May 14/15) Short questions set 2(c)

True. Since $0 \neq \det(\mathbf{AB}) = \det(\mathbf{A}) \cdot \det(\mathbf{B})$; neither $\det(\mathbf{A})$ is zero, nor $\det(\mathbf{B})$. Hence, both are invertible.

Another reasoning, since $\mathbf{AB}$ is invertible, then there exist a matrix $\mathbf{E}$ such that $\mathbf{ABE} = \mathbf{I}$. Hence $\mathbf{A}^{-1} = \mathbf{BE}$.

$\square$

(Final May 14/15) Short questions set 3(a)

We first have to find a vector in the direction of the line. We let

$$\boldsymbol{v} = \begin{pmatrix}2\\4\\1\end{pmatrix} - \begin{pmatrix}1\\3\\1\end{pmatrix} = \begin{pmatrix}1\\1\\0\end{pmatrix}.$$

A parametric representation of the line is therefore

$$\boldsymbol{x} = \boldsymbol{x}_P + a\boldsymbol{v} \quad \Rightarrow \quad \begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix} = \begin{pmatrix}2\\4\\1\end{pmatrix} + a\begin{pmatrix}1\\1\\0\end{pmatrix} \quad \text{or} \quad \begin{cases}x_1 = 2 + a\\ x_2 = 4 + a\\ x_3 = 1\end{cases}.$$

$\square$

(Final May 14/15) Short questions set 3(b)

We need to eliminate the "parametric" part $(a\boldsymbol{v})$. To do that, we need to find two vectors in $\mathbb{R}^3$ orthogonal to $\boldsymbol{v}$. And then we need to multiply the parametric equation by those vectors to find two equations. An easy way to compute all this calculations is by. . . Gaussian Elimination!:

$$\left[\begin{array}{c}[\boldsymbol{v}]^{\intercal}\\ \hline [\boldsymbol{x}]^{\intercal}\\ \hline {\boldsymbol{x}_p}^{\intercal}\end{array}\right] = \left[\begin{array}{ccc}1 & 1 & 0\\ \hline x_1 & x_2 & x_3\\ \hline 2 & 4 & 1\end{array}\right] \xrightarrow{\underset{[(-1)\mathbf{1}+\mathbf{2}]}{\tau}} \left[\begin{array}{ccc}1 & 0 & 0\\ \hline x_1 & x_2 - x_1 & x_3\\ \hline 2 & 2 & 1\end{array}\right] \Rightarrow \begin{cases}x_2 - x_1 & = 2\\ x_3 & = 1\end{cases}.$$

$\square$

(Final May 14/15) Short questions set 4(a)

This set is closed under addition, since for any $\boldsymbol{u}$ and $\boldsymbol{v}$ in $\mathcal{W}$:

$$\begin{pmatrix}u_1\\u_2\\u_3\\bu_1\end{pmatrix} + \begin{pmatrix}v_1\\v_2\\v_3\\bv_1\end{pmatrix} = \begin{pmatrix}u_1+v_1\\u_2+v_2\\u_3+v_3\\bu_1+bv_1\end{pmatrix} = \begin{pmatrix}u_1+v_1\\u_2+v_2\\u_3+v_3\\b(u_1+v_1)\end{pmatrix};$$

and it is closed under scalar multiplication:

$$k\begin{pmatrix}u_1\\u_2\\u_3\\bu_1\end{pmatrix} = \begin{pmatrix}ku_1\\ku_2\\ku_3\\kbu_1\end{pmatrix} = \begin{pmatrix}ku_1\\ku_2\\ku_3\\b(ku_1)\end{pmatrix};$$

it follows that $\mathcal{W}$ is a subspace por any b.

$\square$

(Final May 14/15) Short questions set 4(b)
When $b = 1$, vectors in $\mathcal{W}$ have the form

$$\begin{pmatrix} a \\ b \\ c \\ a \end{pmatrix} = a \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} + b \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + c \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

Hence, dimension is 3 and a basis is:

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}; \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}; \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right\}.$$

$\square$

(Final July 13/14) Exercise 1(a)
The corresponding symmetric matrix is

$$\mathsf{A} = \begin{bmatrix} a & 0 & 0 \\ 0 & 4 & 4 \\ 0 & 4 & -2 \end{bmatrix};$$

and **it is not definite** for any a, since there are positive and negative numbers on the main diagonal; and therefore:

$$\begin{pmatrix} 0 & 1 & 0 \end{pmatrix} \begin{bmatrix} a & 0 & 0 \\ 0 & 4 & 4 \\ 0 & 4 & -2 \end{bmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = 4 > 0; \quad \text{but} \quad \begin{pmatrix} 0 & 0 & 1 \end{pmatrix} \begin{bmatrix} a & 0 & 0 \\ 0 & 4 & 4 \\ 0 & 4 & -2 \end{bmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = -2 < 0.$$

We can also check this by gaussian elimination

$$\begin{bmatrix} a & 0 & 0 \\ 0 & 4 & 4 \\ 0 & 4 & -2 \end{bmatrix} \xrightarrow[{[(-1)\mathbf{2}+\mathbf{3}]}]{\boldsymbol{\tau}} \begin{bmatrix} a & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 4 & -6 \end{bmatrix}$$

(we get positive and negative pivots); or computing the subdeterminants

$$|a| = a; \qquad \begin{vmatrix} a & 0 \\ 0 & 4 \end{vmatrix} = 4a; \qquad \begin{vmatrix} a & 0 & 0 \\ 0 & 4 & 4 \\ 0 & 4 & -2 \end{vmatrix} = a \begin{vmatrix} 4 & 4 \\ 4 & -2 \end{vmatrix} = a(-8-16) = -24a.$$

(we get positive and negative subdeterminants).

$\square$

(Final July 13/14) Exercise 1(b)

$$\det(\mathsf{A} - \boldsymbol{\lambda}\mathsf{I}) = \begin{vmatrix} -\lambda & 0 & 0 \\ 0 & 4-\lambda & 4 \\ 0 & 4 & -2-\lambda \end{vmatrix} = -\lambda \begin{bmatrix} 4-\lambda & 4 \\ 4 & -2-\lambda \end{bmatrix} = -\lambda\left[\lambda^2 - 2\lambda - 24\right] \to \begin{cases} \lambda = 0 \\ \lambda = 6 \\ \lambda = -4 \end{cases}$$

$\square$

(Final July 13/14) Exercise 1(c)
For $\lambda_1 = 0$

$$\mathsf{A} - 0\mathsf{I} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 4 & 4 \\ 0 & 4 & -2 \end{bmatrix} \quad \Rightarrow \quad \boldsymbol{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad \text{is an eigenvector.}$$

For $\lambda_2 = 6$

$$\mathsf{A} - 6\mathsf{I} = \begin{bmatrix} -6 & 0 & 0 \\ 0 & -2 & 4 \\ 0 & 4 & -8 \end{bmatrix} \quad \Rightarrow \quad \boldsymbol{v}_2 = \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} \quad \text{is an eigenvector.}$$

For $\lambda_1 = -4$

$$\mathbf{A} + 4\mathbf{I} = \begin{bmatrix} -4 & 0 & 0 \\ 0 & 8 & 4 \\ 0 & 4 & 2 \end{bmatrix} \quad \Rightarrow \quad \boldsymbol{v}_2 = \begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix} \quad \text{is an eigenvector.}$$

Since each eigenvector is associated to a different eigenvalue, they are linearly independent.

□

(Final July 13/14) Exercise 1(d)

It is easy to check that $\boldsymbol{v}_1$, $\boldsymbol{v}_2$, and $\boldsymbol{v}_3$ are perpendicular:

$$\left[\boldsymbol{v}_1\right]^{\intercal}\left[\boldsymbol{v}_1\right][2]v = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} = 0; \quad \left[\boldsymbol{v}_1\right]^{\intercal}\left[\boldsymbol{v}_1\right][3]v = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix} = 0; \quad \left[\boldsymbol{v}_2\right]^{\intercal}\left[\boldsymbol{v}_2\right][3]v = \begin{pmatrix} 0 & 2 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix} = 0.$$

We need eigenvectors in those directions with unit lenght. Since the lengths are

$$\|\boldsymbol{v}_1\|^2 = \boldsymbol{v}_1 \cdot \boldsymbol{v}_1 = 1, \quad \|\boldsymbol{v}_2\|^2 = \boldsymbol{v}_2 \cdot \boldsymbol{v}_2 = 5, \quad \|\boldsymbol{v}_3\|^2 = \boldsymbol{v}_3 \cdot \boldsymbol{v}_3 = 5;$$

then,

$$\mathbf{Q} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ 0 & \frac{1}{\sqrt{5}} & \frac{-2}{\sqrt{5}} \end{bmatrix}, \quad \text{and} \quad \mathbf{D} = \begin{bmatrix} 0 & & \\ & 6 & \\ & & -4 \end{bmatrix}.$$

□

(Final July 13/14) Exercise 1(e)

No, since this quadratic form is not positive definite.

□

(Final July 13/14) Exercise 2(a)

$$\begin{aligned} \boldsymbol{x}\mathbf{A}^{\intercal}\mathbf{A}\boldsymbol{x} =& (\mathbf{A}\boldsymbol{x}) \cdot (\mathbf{A}\boldsymbol{x}) && \text{product of trasposed matrices} : \mathbf{A}\boldsymbol{x} = \boldsymbol{x}\mathbf{A}^{\intercal} \\ \geq& 0 && \text{the sum of squares of the elements of } \mathbf{A}\boldsymbol{x} \end{aligned}$$

□

(Final July 13/14) Exercise 2(b)

The quadratic form $\boldsymbol{x}(\mathbf{A}^{\intercal}\mathbf{A})\boldsymbol{x}$ is positive definite only when $\mathbf{A}\boldsymbol{x} \neq \mathbf{0}$ for all $\boldsymbol{x} \neq \mathbf{0}$. Therefore, the condition is: *"$\mathbf{A}$ must be full column rank"*, or in other words *"The columns of $\mathbf{A}$ must be linearly independent"*.

□

(Final July 13/14) Exercise 2(c)

If $m < n$, then its columns are linearly dependent and it is possible to find a vector $\boldsymbol{y} \neq \mathbf{0}$ such that $\mathbf{A}\boldsymbol{y} = \mathbf{0}$, and therefore $\boldsymbol{y}(\mathbf{A}^{\intercal}\mathbf{A})\boldsymbol{y} = \left[\mathbf{0}\right]^{\intercal}\left[\mathbf{0}\right] = 0$.

□

(Final July 13/14) Exercise 3(a)

Since the third vector is $\boldsymbol{u} + \boldsymbol{v}$, and obviously $\boldsymbol{u}$ and $\boldsymbol{v}$ are linearly independent, then any two of them is a basis, for example, $\boldsymbol{u}$ and $\boldsymbol{v}$.

□

(Final July 13/14) Exercise 3(b)

We are asked to solve $x\boldsymbol{u} + y\boldsymbol{v} = (1, 0, -1, 1)^{\intercal}$.

$$\left[\begin{array}{cc|c} 2 & 1 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \\ 2 & 1 & -1 \\ \hline 1 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right] \xrightarrow[(1)\mathbf{2}+\mathbf{3}]{\substack{\boldsymbol{\tau} \\ [(-2)\mathbf{2}+\mathbf{1}]}} \left[\begin{array}{cc|c} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \\ \hline 1 & 0 & 0 \\ -2 & 1 & 1 \end{array}\right] \xrightarrow{\substack{\boldsymbol{\tau} \\ [(-1)\mathbf{1}+\mathbf{3}]}} \left[\begin{array}{cc|c} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ \hline 1 & 0 & -1 \\ -2 & 1 & 3 \end{array}\right]$$

Hence, the vector belongs to $\mathcal{S}$, and $x = -1$ and $y = 3$; so the coordinates with respect to the basis in part (a) are $(-1, 3)$.

□

(Final July 13/14) Exercise 3(c)

Since:

$$\mathcal{S} = \left\{ \boldsymbol{x} \in \mathbb{R}^4 \quad \text{such that} \quad \boldsymbol{x} = a\boldsymbol{u} + b\boldsymbol{v} \right\},$$

we need to multiply the parametric equation $\boldsymbol{x} = a\boldsymbol{u} + b\boldsymbol{v}$ by two vectors in $\mathbb{R}^4$ perpendicular to $\boldsymbol{u}$ and $\boldsymbol{v}$, then the parametric part will disappear. We can do this by gaussian column elimination if we write a matrix whos rows are $\boldsymbol{u}$, $\boldsymbol{v}$ and $\boldsymbol{x}$, and we get a column of zeros on the rows corresponding to $\boldsymbol{u}$ y $\boldsymbol{v}$:

$$\left[\, \boldsymbol{u} \quad \boldsymbol{v} \,\middle|\, \boldsymbol{x} \,\middle|\, \mathbf{I} \,\right]^{\intercal} = \left[\begin{array}{cccc} 2 & 0 & 1 & 2 \\ 1 & 0 & 0 & 1 \\ \hline x & y & z & t \\ \hline 1 & 0 & & \\ & 1 & & \\ & 0 & 1 & \\ & 0 & & 1 \end{array}\right] \xrightarrow{\underset{[(-1)\mathbf{1}+\mathbf{4}]}{\boldsymbol{\tau}}} \left[\begin{array}{cccc} 2 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ \hline x & y & z & t-x \\ \hline 1 & 0 & & -1 \\ & 1 & & 0 \\ & 0 & 1 & 0 \\ & 0 & & 1 \end{array}\right];$$

and therefore the cartesian equations are:

$$\begin{cases} y & = 0 \\ t - x & = 0 \end{cases}$$

□

(Final July 13/14) Exercise 3(d)

We have multiply the parametric equations $\boldsymbol{x} = a\boldsymbol{u} + b\boldsymbol{v}$ by two vectors to get the implicit equations, those vectors are a basis of the orthogonal complement of $\mathcal{S}$

$$\text{Basis for } \mathcal{S}^{\perp} = \left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}; \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

.

□

(Final July 13/14) Exercise 3(e)

Since $\dim \mathcal{S}$ is 2, we just only need to find choose one vector that is not linear combination of $\boldsymbol{u}$,$\boldsymbol{v}$ and $\boldsymbol{w}$. Any vector in $\mathcal{S}^{\perp}$ is perpendicular to $\boldsymbol{u}$, $\boldsymbol{v}$ and $\boldsymbol{w}$.

Hence, any linear combination of vectors in the basis of part (d) is a good answer, for example:

$$\boldsymbol{z} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}.$$

□

(Final July 13/14) Short questions set 1(a)

True: since the inverse of a squared matrix is unique, and since $\mathbf{AA} = \mathbf{I}$, it follows that matrix $\mathbf{A}$ is its own inverse.

□

(Final July 13/14) Short questions set 1(b)

True: Any matrix is orthonormal when has perpendicular columns of length one; therefore, a matrix $\mathbf{A}$ is orthonormal if and only if $\mathbf{A}^{\intercal}\mathbf{A} = \mathbf{I}$. Hence, when $\mathbf{A}$ is symmetric

$$\mathbf{I} = \mathbf{A}^2 = \mathbf{AA} = \mathbf{A}^{\intercal}\mathbf{A} \quad \Rightarrow \quad \text{Matrix } \mathbf{A} \text{ is orthonormal.}$$

□

(Final July 13/14) Short questions set 1(c)

False. The null matrix is a counterexample, since $\mathbf{0}^2 = \mathbf{0}$; and $\operatorname{rg}(\mathbf{0}) = 0$.

□

(Final July 13/14) Short questions set 1(d)

False. From part (c) we known that matrix $\mathbf{B}$ could be singular (and then there is no $\mathbf{B}^{-1}$). Hence, since there is no justification for the use of $\mathbf{B}^{-1}$, the deduction is false.

$\square$

(Final July 13/14) Short questions set 2(a)

True. If 0 is an eigenvalue, then $\det \mathbf{A}$ is zero, and therefore the matrix is singular.

$\square$

(Final July 13/14) Short questions set 2(b)

True: That -3 is an eigenvalue means that the nullspace of $(\mathbf{A}+3\mathbf{I})$ is nontrivial (dimension >0) so, as $\dim(\textit{null space}) + \dim(\textit{column space}) = n$, one must have $\operatorname{rg}\left(\mathbf{A}+3\mathbf{I}\right) < n$: there must be vectors $\boldsymbol{v}$ not in $\mathcal{C}(\mathbf{A}+3\mathbf{I})$.

$\square$

(Final July 13/14) Short questions set 3(a)

For $a = 1, -1, 2$, since for those values the matrix is singular (for $a=1$ the first and last columns are equal, for $a=-1$ the second and last columns are equal, for $a=2$ the third and last columns are equal).

$\square$

(Final July 13/14) Short questions set 3(b)

By gaussian row elimination we get:

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 2 & a \\ 1 & 1 & 4 & a^2 \\ 1 & -1 & 8 & a^3 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & -2 & 1 & a-1 \\ 0 & 0 & 3 & a^2-1 \\ 0 & -2 & 7 & a^3-1 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & -2 & 1 & a-1 \\ 0 & 0 & 3 & a^2-1 \\ 0 & 0 & 6 & a^3-a \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & -2 & 1 & a-1 \\ 0 & 0 & 3 & (a-1)(a+1) \\ 0 & 0 & 6 & a(a-1)(a+1) \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & -2 & 1 & a-1 \\ 0 & 0 & 3 & (a-1)(a+1) \\ 0 & 0 & 0 & a(a-1)(a+1)-2(a-1)(a+1) \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & -2 & 1 & a-1 \\ 0 & 0 & 3 & (a-1)(a+1) \\ 0 & 0 & 0 & (a-2)(a-1)(a+1) \end{vmatrix}$$
$$= -6(a-2)(a-1)(a+1).$$

We can get the same by gaussian column elimination:

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 2 & a \\ 1 & 1 & 4 & a^2 \\ 1 & -1 & 8 & a^3 \end{vmatrix} = \begin{vmatrix} 1 & & & \\ 1 & -2 & 1 & a-1 \\ 1 & 0 & 3 & a^2-1 \\ 1 & -2 & 7 & a^3-1 \end{vmatrix} = \begin{vmatrix} 1 & & & \\ 1 & -2 & 1 & a-1 \\ 1 & 0 & 3 & (a-1)(1+a) \\ 1 & -2 & 7 & a^3-1 \end{vmatrix} = \begin{vmatrix} 1 & & & \\ 1 & -2 & & \\ 1 & 0 & 3 & (a-1)(1+a) \\ 1 & -2 & 6 & a^3-a \end{vmatrix}$$

$$= \begin{vmatrix} 1 & & & \\ 1 & -2 & & \\ 1 & 0 & 3 & (a-1)(1+a) \\ 1 & -2 & 6 & a(a-1)(1+a) \end{vmatrix} = \begin{vmatrix} 1 & & & \\ 1 & -2 & & \\ 1 & 0 & 3 & \\ 1 & -2 & 6 & (a-2)(a-1)(1+a) \end{vmatrix} = -6(a-2)(a-1)(a+1). \quad (1)$$

$\square$

(Final July 13/14) Short questions set 3(c)

When a is 1, -1 or 2 the matrix is singular (rank 3), hence $\dim\mathcal{N}\left(\mathbf{A}\right) = 1$. Otherwise the matrix is full rank so $\dim\mathcal{N}\left(\mathbf{A}\right) = 0$.

$\square$

(Final July 13/14) Short questions set 3(d)

Since $\mathbf{M}_a$ is full rank when $a=0$, there is only one solution: $\boldsymbol{x} = \mathbf{0}$.

$\square$

(Final May 13/14) Exercise 1(a)

1. Since $\det(\mathbf{A}) = 3-a$, matrix $\mathbf{A}$ is invertible when $a \neq 3$.
2. $\mathbf{A}$ is symmetric for any a.
3. Since $\mathbf{A}$ is symmetric, it is digonalizable for any a.

$\square$

(Final May 13/14) Exercise 1(b)

Matrix $\mathbf{A}$ is never definite. We can check this using the subdeterminants test

- first subdeterminant $= 1 > 0$,

- second subdeterminant $= -1 < 0$,

$\square$

(Final May 13/14) Exercise 1(c)

Any solution to $(\mathbf{A} - 0\mathbf{I})\boldsymbol{x} = \mathbf{A}\boldsymbol{x} = \mathbf{0}$ is an eigenvector for $\lambda = 0$. By gaussian column reduction we get

$$\left[\begin{array}{ccc} 1 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & a \\ \hline 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right] \xrightarrow{\substack{\boldsymbol{\tau} \\ [(-1)\mathbf{1}+\mathbf{2}] \\ (-2)\mathbf{1}+\mathbf{3}}} \left[\begin{array}{ccc} 1 & 0 & 0 \\ 1 & -1 & -1 \\ 2 & -1 & a-4 \\ \hline 1 & -1 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right] \xrightarrow{\substack{\boldsymbol{\tau} \\ [(-1)\mathbf{2}+\mathbf{3}]}} \left[\begin{array}{ccc} 1 & 0 & 0 \\ 1 & -1 & 0 \\ 2 & -1 & a-3 \\ \hline 1 & -1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{array}\right].$$

When $a = 3$, the matrix is singular, so $\det(\mathbf{A}) = 0$ and $\lambda = 0$ is an eigenvalue of $\mathbf{A}$ with a corresponding eigenvector $\boldsymbol{x} = \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$.

$\square$

(Final May 13/14) Exercise 1(d)

Since for any squared matrix

$$\mathbf{A}\boldsymbol{v} = \boldsymbol{v}\lambda \quad \Longrightarrow \quad \mathbf{A}^2\boldsymbol{v} = \mathbf{A}\boldsymbol{v}\lambda = \boldsymbol{v}\lambda^2,$$

the square of the eigenvalues of $\mathbf{A}$ are eigenvalues of $\mathbf{A}^2$ with the same corresponding eigenvectors, hence $\lambda = 0$ and $\boldsymbol{v} = \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$ are respectively an eigenvalue and an eigenvector of $\mathbf{A}^2$.

$\square$

(Final May 13/14) Exercise 1(e)

When $a = 3$, the third column is a linear combination of the two first (and the two first columns are independent), hence, the rank is 2 and so it is $\dim \mathcal{C}\left(\mathbf{A}\right)$.

The following are parametric equations for $\mathcal{C}\left(\mathbf{A}\right)$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = a \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} + b \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}.$$

We must get rid of the parametric part if we want get the implicit equations of $\mathcal{C}\left(\mathbf{A}\right)$. So we have to multiply the parametric equations by a vector orthogonal to $\mathcal{C}\left(\mathbf{A}\right)$.

In part c) we found $\begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$, a vector perpendicular to the row space of $\mathbf{A}$ but, since $\mathbf{A} = \mathbf{A}^{\intercal}$, the column space $\mathcal{C}\left(\mathbf{A}\right)$ equals the row space $\mathcal{C}\left(\mathbf{A}^{\intercal}\right)$ and therefore, that vector is also perpendicular to $\mathcal{C}\left(\mathbf{A}\right)$. Hence

$$\begin{pmatrix} -1 & -1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = a \begin{pmatrix} -1 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} + b \begin{pmatrix} -1 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \Longrightarrow \boxed{-x - y + z = 0}$$

The same answer we get by gaussian column reduction:

$$\left[\begin{array}{ccc} 1 & 1 & 2 \\ 1 & 0 & 1 \\ \hline x & y & z \end{array}\right] \xrightarrow{\substack{\boldsymbol{\tau} \\ [(-1)\mathbf{1}+\mathbf{2}] \\ (-2)\mathbf{1}+\mathbf{3}}} \left[\begin{array}{ccc} 1 & 0 & 0 \\ 1 & -1 & -1 \\ \hline x & y-x & z-2x \end{array}\right] \xrightarrow{\substack{\boldsymbol{\tau} \\ [(-1)\mathbf{2}+\mathbf{3}]}} \left[\begin{array}{ccc} 1 & 0 & 0 \\ 1 & -1 & 0 \\ \hline x & y-x & z-y-x \end{array}\right]$$

Therefore, the implicit equation of $\mathcal{C}\left(\mathbf{A}\right)$ is $\{z - y - x = 0$.

$\square$

(Final May 13/14) Exercise 2(a)

$$\left[\begin{array}{cccc|c} 1 & 2 & 0 & m & -2 \\ 0 & 1 & -1 & 2 & -2 \\ 1 & 2 & 0 & 0 & -n \\ 2 & 4 & 1 & 3 & -2 \\ \hline 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right] \xrightarrow{\substack{\boldsymbol{\tau} \\ [(-2)\mathbf{1}+\mathbf{2}] \\ (-m)\mathbf{1}+\mathbf{4} \\ (2)\mathbf{1}+\mathbf{5}}} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 2 & -2 \\ 1 & 0 & 0 & -m & 2-n \\ 2 & 0 & 1 & 3-2m & 2 \\ \hline 1 & -2 & 0 & -m & 2 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right] \xrightarrow{\substack{\boldsymbol{\tau} \\ [(1)\mathbf{2}+\mathbf{3}] \\ (-2)\mathbf{2}+\mathbf{4} \\ (2)\mathbf{2}+\mathbf{5}}} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & -m & 2-n \\ 2 & 0 & 1 & 3-2m & 2 \\ \hline 1 & -2 & -2 & 4-m & -2 \\ 0 & 1 & 1 & -2 & 2 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right] \xrightarrow{\substack{\boldsymbol{\tau} \\ [\mathbf{3}\rightleftharpoons\mathbf{4}] \\ (-2)\mathbf{4}+\mathbf{5}}} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & -m & 0 & 2-n \\ 2 & 0 & 3-2m & 1 & 0 \\ \hline 1 & -2 & 4-m & -2 & 2 \\ 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 1 & 0 & 0 \end{array}\right]$$

$\square$

If $m = 0$ the rank is 3, otherwise the rank is 4.

□

(Final May 13/14) Exercise 2(b)
When $m \neq 0$, the system has a single unique solution ($\operatorname{rg}(\mathbf{A}) = 4$).
When $m = 0$, two different cases are possible:

- If $n \neq 2$ the system is not solvable
- If $n = 2$ the system has infinitely many solutions.

□

(Final May 13/14) Exercise 2(c)

$$\left[\begin{array}{cccc|c} 1&2&0&0&-2\\ 0&1&-1&2&-2\\ 1&2&0&0&-2\\ 2&4&1&3&-2\\ \hline 1&0&0&0&0\\ 0&1&0&0&0\\ 0&0&1&0&0\\ 0&0&0&1&0 \end{array}\right] \xrightarrow[(2)\mathbf{1}+\mathbf{5}]{\substack{\tau\\ [(-2)\mathbf{1}+\mathbf{2}]}} \left[\begin{array}{cccc|c} 1&0&0&0&0\\ 0&1&-1&2&-2\\ 1&0&0&0&0\\ 2&0&1&3&2\\ \hline 1&-2&0&0&2\\ 0&1&0&0&0\\ 0&0&1&0&0\\ 0&0&0&1&0 \end{array}\right] \xrightarrow[(2)\mathbf{2}+\mathbf{5}]{\substack{\tau\\ [(1)\mathbf{2}+\mathbf{3}]\\ (-2)\mathbf{2}+\mathbf{4}}} \left[\begin{array}{cccc|c} 1&0&0&0&0\\ 0&1&0&0&0\\ 1&0&0&0&0\\ 2&0&1&3&2\\ \hline 1&-2&-2&4&-2\\ 0&1&1&-2&2\\ 0&0&1&0&0\\ 0&0&0&1&0 \end{array}\right] \xrightarrow[(-2)\mathbf{3}+\mathbf{5}]{\substack{\tau\\ [(-3)\mathbf{3}+\mathbf{4}]}} \left[\begin{array}{cccc|c} 1&0&0&0&0\\ 0&1&0&0&0\\ 1&0&0&0&0\\ 2&0&1&0&0\\ \hline 1&-2&-2&10&2\\ 0&1&1&-5&0\\ 0&0&1&-3&-2\\ 0&0&0&1&0 \end{array}\right]$$

In this case the set of solutions is $\left\{ \boldsymbol{x} \in \mathbb{R}^4 \text{ such that } \boldsymbol{x} = \begin{pmatrix} 2\\0\\-2\\0 \end{pmatrix} + a \begin{pmatrix} 10\\-5\\-3\\1 \end{pmatrix}; \quad a \in \mathbb{R} \right\}$.

□

(Final May 13/14) Exercise 2(d)
Since $\operatorname{rg}(A) \geq 3$, there will never be two special solutions. So the set of solution will never be a plane.

□

(Final May 13/14) Exercise 2(e)

$$\begin{vmatrix} 1&2&0&m\\ 0&1&-1&2\\ 1&2&0&0\\ 2&4&1&3 \end{vmatrix} = \begin{vmatrix} 1&-1&2\\ 2&0&0\\ 4&1&3 \end{vmatrix} + \begin{vmatrix} 2&0&m\\ 1&-1&2\\ 4&1&3 \end{vmatrix} - 2\begin{vmatrix} 2&0&m\\ 1&-1&2\\ 2&0&0 \end{vmatrix} = (10) + (5m-10) - 2(2m) = m.$$

□

(Final May 13/14) Exercise 3(a)

$$\left[\begin{array}{c|c} \mathbf{A} & -\boldsymbol{b}\\ \hline \mathbf{I} & \mathbf{0} \end{array}\right] = \left[\begin{array}{ccc|c} 1&0&-2&-2\\ 0&3&0&-3\\ -2&0&6&-1\\ \hline 1&0&0&0\\ 0&1&0&0\\ 0&0&1&0 \end{array}\right] \xrightarrow{\substack{\tau\\ [(2)\mathbf{1}+\mathbf{3}]\\ (2)\mathbf{1}+\mathbf{4}\\ (1)\mathbf{2}+\mathbf{4}}} \left[\begin{array}{ccc|c} 1&0&0&0\\ 0&3&0&0\\ -2&0&2&-5\\ \hline 1&0&2&2\\ 0&1&0&1\\ 0&0&1&0 \end{array}\right] \xrightarrow{\substack{\tau\\ [(5/2)\mathbf{3}+\mathbf{4}]}} \left[\begin{array}{ccc|c} 1&0&0&0\\ 0&3&0&0\\ -2&0&2&0\\ \hline 1&0&2&7\\ 0&1&0&1\\ 0&0&1&5/2 \end{array}\right] = \left[\begin{array}{c|c} \mathbf{L} & \mathbf{0}\\ \hline \mathbf{E} & \boldsymbol{x}_p \end{array}\right]$$

The system $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$ has the following single unique solution, $\boldsymbol{x}_p = \begin{pmatrix} 7\\1\\5/2 \end{pmatrix}$.

□

(Final May 13/14) Exercise 3(b)
Since the matrix is symmetric, it is possible to get the following $\dot{\mathbf{L}}\mathbf{D}\dot{\mathbf{U}}$ factorization:

$$\mathbf{A} = \dot{\mathbf{L}}\mathbf{D}\dot{\mathbf{U}} = \dot{\mathbf{U}}^{\intercal}\mathbf{D}\dot{\mathbf{U}} = \begin{bmatrix} 1&0&0\\ 0&1&0\\ -2&0&1 \end{bmatrix} \begin{bmatrix} 1&&\\ &3&\\ &&2 \end{bmatrix} \begin{bmatrix} 1&0&-2\\ 0&1&0\\ 0&0&1 \end{bmatrix}$$

where $\dot{\mathbf{U}} = \mathbf{E}^{-1}$ and $\dot{\mathbf{L}} = \dot{\mathbf{U}}^{\mathsf{T}}$. Therefore, the quadratic form $\boldsymbol{x}\mathbf{A}\boldsymbol{x} = \boldsymbol{x}\dot{\mathbf{L}}\mathbf{D}\dot{\mathbf{U}}\boldsymbol{x}$ can be written as

$$\boldsymbol{x}\mathbf{A}\boldsymbol{x} = \boldsymbol{x}\dot{\mathbf{L}}\mathbf{D}\dot{\mathbf{U}}\boldsymbol{x} = \boldsymbol{x}\begin{bmatrix}1&0&0\\0&1&0\\-2&0&1\end{bmatrix}\begin{bmatrix}1&&\\&3&\\&&2\end{bmatrix}\begin{bmatrix}1&0&-2\\0&1&0\\0&0&1\end{bmatrix}\boldsymbol{x} = 1(x-2z)^2+3y^2+2z^2>0;$$

where $\boldsymbol{x} = \begin{bmatrix}x & y & z\end{bmatrix}$, so it is positive definite since all pivots (1,3,2) are positive.

□

(Final May 13/14) Exercise 3(c)

$|\mathbf{A}| = 6 \neq 0$ (product of the pivots $= |\mathbf{A}|$), therefore, zero can not be an eigenvalue of $\mathbf{A}$ (it is a full rank matrix).

□

(Final May 13/14) Exercise 3(d)

$\mathbf{A}\boldsymbol{v} = \begin{bmatrix}1&0&-2\\0&3&0\\-2&0&6\end{bmatrix}\begin{pmatrix}0\\4\\0\end{pmatrix} = \begin{pmatrix}0\\12\\0\end{pmatrix} = 3\boldsymbol{v}$. Hence, $\boldsymbol{v}$ is an eigenvetor, and $\lambda = 3$ is the corresponding eigenvalue.

□

(Final May 13/14) Short questions set 1.

The parametric equations are

$$\boldsymbol{x} = \begin{pmatrix}0\\0\\0\\1\end{pmatrix} + a\begin{pmatrix}1\\1\\0\\1\end{pmatrix} + b\begin{pmatrix}0\\0\\1\\1\end{pmatrix}$$

Again, we must get rid of the parametric part if we want get the implicit equations. So we have to multiply the parametric equations by a basis of the orthogomal complement of that plane. We can do that by gaussian elimination:

$$\left[\begin{array}{cccc}1&1&0&1\\0&0&1&1\\\hline x&y&z&w\\\hline 0&0&0&1\end{array}\right] \xrightarrow[(-1)\mathbf{1}+\mathbf{4}]{\substack{\tau\\ [(-1)\mathbf{1}+\mathbf{2}]}} \left[\begin{array}{cccc}1&0&0&0\\0&0&1&1\\\hline x&(y-x)&z&(w-x)\\\hline 0&0&0&1\end{array}\right] \xrightarrow{\substack{\tau\\ [(-1)\mathbf{3}+\mathbf{4}]}} \left[\begin{array}{cccc}1&0&0&0\\0&0&1&0\\\hline x&(y-x)&z&(w-x-z)\\\hline 0&0&0&1\end{array}\right]$$

Hence, the implicit (or cartesian) equations of this plane are

$$\begin{cases}y-x & =0\\ w-x-z & =1\end{cases}$$

□

(Final May 13/14) Short questions set 2(a)

True: If $\mathbf{A}^2 = \mathbf{I}$, then $|\mathbf{A}^2| = |\mathbf{A}|\cdot|\mathbf{A}| = |\mathbf{I}| = 1$; hence, $|\mathbf{A}| \neq 0$, and therefore $\mathbf{A}$ is full rank matrix.

□

(Final May 13/14) Short questions set 2(b)

False: For example, $\mathbf{0}^2 = \mathbf{0}$, or $\begin{bmatrix}1&0\\0&0\end{bmatrix}$, or any projection matrix $\mathbf{A}\left(\mathbf{A}^{\mathsf{T}}\mathbf{A}\right)^{-1}\mathbf{A}^{\mathsf{T}}$. The matrices $\mathbf{B}$ such that $\mathbf{B}^2 = \mathbf{B}$ are call *idempotent* matrices.

□

(Final May 13/14) Short questions set 2(c)

False: If $\lambda = 0$ is an eigenvalue of $\mathbf{A}$ the matrix is singular. Hence, the columns are dependent and therefore, there are vectors $\boldsymbol{x} \neq \mathbf{0}$ such that $\mathbf{A}\boldsymbol{x} = \mathbf{0}$.

□

(Final May 13/14) Short questions set 3.

The following equations

$$\begin{cases}3x+2y-z & =0\\ 2y+4z & =0\end{cases},$$

are the implicit equations of $\mathcal{W}$. This subspace is the set of solutions to the implicit equations, so we need a set of linearly independ solutions to the system that spans the whole subspace:

$$\left[\begin{array}{rrr} 3 & 2 & -1 \\ 0 & 2 & 4 \\ \hline 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right] \xrightarrow[(3)\mathbf{3}+\mathbf{1}]{\underset{[(2)\mathbf{3}+\mathbf{2}]}{\tau}} \left[\begin{array}{rrr} 0 & 0 & -1 \\ 12 & 10 & 4 \\ \hline 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 2 & 1 \end{array}\right] \xrightarrow[\underset{[(12)\mathbf{2}]}{\tau}]{\underset{[(10)\mathbf{1}]}{\tau}} \left[\begin{array}{rrr} 0 & 0 & -1 \\ 120 & 120 & 4 \\ \hline 10 & 0 & 0 \\ 0 & 12 & 0 \\ 30 & 24 & 1 \end{array}\right] \xrightarrow{\underset{[(-1)\mathbf{2}+\mathbf{1}]}{\tau}} \left[\begin{array}{rrr} 0 & 0 & -1 \\ 0 & 120 & 4 \\ \hline 10 & 0 & 0 \\ -12 & 12 & 0 \\ 6 & 24 & 1 \end{array}\right]$$

Hence, the following set is basis for $\mathcal{W}$: $\left\{ \begin{pmatrix} 10 \\ -12 \\ 6 \end{pmatrix} \right\}$.

□

(Final May 13/14) Short questions set 4(a)

- $x > 0$
- $xy - \frac{1}{4} > 0 \;\Rightarrow\; y > \frac{1}{4x}$

□

(Final May 13/14) Short questions set 4(b)

- Columns are linearly perpendicular when: $\frac{x}{2} + \frac{y}{2} = 0 \;\Rightarrow\; x = -y$.
- The lenght of the columns is one when: $\sqrt{x^2 + \frac{1}{4}} = 1 \;\Rightarrow\; x = \pm\sqrt{3/4}$.

□

(Final May 13/14) Short questions set 5(a)

True:

$$\det(\mathbf{A}^n) = \det(\underbrace{\mathbf{A}\cdots\mathbf{A}}_{n \text{ times}}) = \underbrace{\det(\mathbf{A})\cdots\det(\mathbf{A})}_{n \text{ times}} = \Big(\det(\mathbf{A})\Big)^n = (-1)^n.$$

□

(Final May 13/14) Short questions set 5(b)

True: If $\mathbf{A}$ is idempotent then $\mathbf{A}^2 = \mathbf{A}$; so $\det(\mathbf{A})^2 = \det(\mathbf{A})$. But this is only possible if $\det(\mathbf{A})$ is one or zero.

□

(Final May 13/14) Short questions set 5(c)

True: When $\mathbf{A}$ is positive definite, first entry a_{11} and $\det(\mathbf{A})$ are positive.
When $\mathbf{A}$ is negative definite, first entry a_{11} is negative, but $\det(\mathbf{A})$ is positive.
Therefore, this matrix is not definite.

□

(Final July 12/13) Exercise 1(c)

$$\left[\begin{array}{rrrrr|r} 1 & 1 & 2 & 0 & -1 & -1 \\ 2 & 3 & 3 & -1 & a & -3 \\ 1 & 2 & 1 & -1 & 1 & -2 \\ \hline 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right] \xrightarrow{\substack{\underset{[(-1)\mathbf{1}+\mathbf{2}]}{\tau} \\ (-2)\mathbf{1}+\mathbf{3} \\ (1)\mathbf{1}+\mathbf{5} \\ (1)\mathbf{1}+\mathbf{6}}} \left[\begin{array}{rrrrr|r} 1 & 0 & 0 & 0 & 0 & 0 \\ 2 & 1 & -1 & -1 & a+2 & -1 \\ 1 & 1 & -1 & -1 & 2 & -1 \\ \hline 1 & -1 & -2 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right] \xrightarrow{\substack{\underset{[(1)\mathbf{2}+\mathbf{3}]}{\tau} \\ (1)\mathbf{2}+\mathbf{4} \\ (-a-2)\mathbf{2}+\mathbf{5} \\ (1)\mathbf{2}+\mathbf{6}}} \left[\begin{array}{rrrrr|r} 1 & 0 & 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & -a & 0 \\ \hline 1 & -1 & -3 & -1 & a+3 & 0 \\ 0 & 1 & 1 & 1 & -a-2 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

□

(Final July 12/13) Exercise 1(a)

Any of the following answers is correct:

- If you compute by columns, then the column echelon form is: $\left[\begin{array}{rrrrr|r} 1 & 0 & 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & -a & 0 \end{array}\right]$.

- If you compute by rows, the row echelon form is:

$$\left[\begin{array}{ccccc|c} 1 & 1 & 2 & 0 & -1 & 1 \\ 2 & 3 & 3 & -1 & a & 3 \\ 1 & 2 & 1 & -1 & 1 & 2 \end{array}\right] \xrightarrow[(-1)\mathbf{3}+\mathbf{1}]{\substack{\tau \\ [(-2)\mathbf{2}+\mathbf{1}]}} \left[\begin{array}{ccccc|c} 1 & 1 & 2 & 0 & -1 & 1 \\ 0 & 1 & -1 & -1 & a+2 & 1 \\ 0 & 1 & -1 & -1 & 2 & 1 \end{array}\right] \xrightarrow{\substack{\tau \\ [(-1)\mathbf{3}+\mathbf{2}]}} \left[\begin{array}{ccccc|c} 1 & 1 & 2 & 0 & -1 & 1 \\ 0 & 1 & -1 & -1 & a+2 & 1 \\ 0 & 0 & 0 & 0 & -a & 0 \end{array}\right]$$

□

(Final July 12/13) Exercise 1(b)

The system is consistent for any value of a (note that $\boldsymbol{b}$ equals the second column of $\mathbf{A}$).

When $a = 0$ the rank of $\mathbf{A}$ is two, and then dimension of $\mathcal{N}(\mathbf{A})$ is three (three free columns); and therefore, *the set of solutions is a three dimensional hyperplane in* $\mathbb{R}^5$.

When $a \neq 0$ the rank of $\mathbf{A}$ is three, and then dimension of $\mathcal{N}(\mathbf{A})$ is two (only two free columns); and therefore, *the set of solutions is a plane in* $\mathbb{R}^5$.

□

(Final July 12/13) Exercise 1(c)

When $a = 1$ the rank of $\mathbf{A}$ is three, so only three variables can be consider as pivot (or dependent or endogenous) variables.

(*In the next paragraphs we will refer to gaussian elimination by columns, but it is possible to reach the same conclussion using elimination by rows, or using subdeterminants*)

After the gaussian elimination process the last column has a pivot, and thus this column is linearly independent; therefore x_5 is a pivot (or dependent or endogenous) variable. Let's consider x_5 as the third pivot variable and let's find the other two... Consider the submatrix with the first four columns of A. Any of the first three columns can be taken as pivot; since its first components are non-zero. It is easy to check that after the gaussian elimination process (using any of the first three columns as pivot) the second components of the remaining columns are non-zero, and therefore, any of them can be choosen as second pivot column. Thus, x_5 *is always a pivot variable, but we can choose any two of the remaining variables as pivots variables*.

□

(Final July 12/13) Exercise 1(c)

The dimension is two, because there are only two columns of zeros in the coefficient matrix after Gaussian elimination, and it is easy to see that a basis of the set of solutions to $\mathbf{A}\boldsymbol{x} = \mathbf{0}$ is formed by the two vectors appearing below the columns of zeros:

$$\text{Basis:} \qquad \left\{ \begin{pmatrix} -3 \\ 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}; \begin{pmatrix} -1 \\ 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right\}.$$

□

(Final July 12/13) Exercise 1(c)

A particular solution appears below the last column of zeros corresponding to the right hand side vector of the system; thus, the set of vectors $\boldsymbol{x}$ that verifies $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$ is:

$$\left\{ \text{the set of vectors } \boldsymbol{x} \text{ in } \mathbb{R}^5 \text{ such that } \boldsymbol{x} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + p \begin{pmatrix} -3 \\ 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + q \begin{pmatrix} -1 \\ 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \text{ for all } p, q \in \mathbb{R} \right\}.$$

□

(Final July 12/13) Exercise 2(a)

Matrix $\mathbf{B}$ is symmetric and therefore diagonalizable. Matrix $\mathbf{C}$ is upper triangular and hence the diagonal elements are the eigenvalues and, since the eigenvalues are not repeated, this matrix is also diagonalizable. It remains to analyze the matrix $\mathbf{A}$. Computing the characteristic polynomial $|\mathbf{A} - \lambda\mathbf{I}| = 0$ we get:

$$\begin{vmatrix} 1-\lambda & 2 & b \\ 0 & -1-\lambda & -3 \\ 0 & 2 & 4-\lambda \end{vmatrix} = (1-\lambda)(\lambda^2 - 3\lambda + 2) = 0 \quad \rightarrow \quad \begin{cases} \lambda = 1 & \text{(double)} \\ \lambda = 2 \end{cases}.$$

Since we have a double eigenvalue, the matrix is diagonalizable only if the eigenspace associated to that eigenvalue is two-dimensional, i.e., only if the rank of $(\mathbf{C}-\mathbf{I}) = \begin{bmatrix} 0 & 2 & b \\ 0 & -2 & -3 \\ 0 & 2 & 3 \end{bmatrix}$ is 1. Hence, $\mathbf{C}$ is diagonalizable only when $b = 3$ (since the third column is a multiple of the second in this case).

□

(Final July 12/13) Exercise 2(b)

This is only possible for symmetric matrices, hence it is only possible for $\mathbf{B}$.

□

(Final July 12/13) Exercise 2(c)

The eigenvalues of the $\mathbf{A}^{-1}$ are the inverse of the eigenvalues of $\mathbf{A}$, so the eigenvalues of $\mathbf{A}^{-1}$ are $\lambda = 1$ (double) and $\lambda = \frac{1}{2}$.

The eigenvectors of $\mathbf{A}^{-1}$ and $\mathbf{A}$ are the same; hence, if $\mathbf{A}$ is diagonalizable then so is $\mathbf{A}^{-1}$ (In part (a) we have seen that $\mathbf{A}$ is diagonalizable only when $b = 3$). We can find the eigenvectors of $\mathbf{A}^{-1}$ computing those of $\mathbf{A}$ (since they are the same). For $\lambda = 1$ (double):

$$\left[\begin{array}{c} \mathbf{A}-1\mathbf{I} \\ \hline \mathbf{I} \end{array}\right] = \left[\begin{array}{ccc} 0 & 2 & 3 \\ 0 & -2 & -3 \\ 0 & 2 & 3 \\ \hline 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right] \xrightarrow{\substack{[(3)\overset{\tau}{\mathbf{2}}] \\ (2)\mathbf{3}}} \left[\begin{array}{ccc} 0 & 6 & 6 \\ 0 & -6 & -6 \\ 0 & 6 & 6 \\ \hline 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{array}\right] \xrightarrow{[(-1)\overset{\tau}{\mathbf{2}}+\mathbf{3}]} \left[\begin{array}{ccc} 0 & 6 & 0 \\ 0 & -6 & 0 \\ 0 & 6 & 0 \\ \hline 1 & 0 & 0 \\ 0 & 3 & -3 \\ 0 & 0 & 2 \end{array}\right] \xrightarrow{[\mathbf{1}\overset{\tau}{\rightleftharpoons}\mathbf{2}]} \left[\begin{array}{ccc} 6 & 0 & 0 \\ -6 & 0 & 0 \\ 6 & 0 & 0 \\ \hline 0 & 1 & 0 \\ 3 & 0 & -3 \\ 0 & 0 & 2 \end{array}\right] = \left[\begin{array}{c} \mathbf{L} \\ \hline \mathbf{E} \end{array}\right],$$

and for $\lambda = 2$:

$$\left[\begin{array}{c} \mathbf{A}-2\mathbf{I} \\ \hline \mathbf{I} \end{array}\right] = \left[\begin{array}{ccc} -1 & 2 & 3 \\ 0 & -3 & -3 \\ 0 & 2 & 2 \\ \hline 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right] \xrightarrow{\substack{[(2)\overset{\tau}{\mathbf{1}}+\mathbf{2}] \\ (3)\mathbf{1}+\mathbf{3}}} \left[\begin{array}{ccc} -1 & 0 & 0 \\ 0 & -3 & -3 \\ 0 & 2 & 2 \\ \hline 1 & 2 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right] \xrightarrow{[(-1)\overset{\tau}{\mathbf{2}}+\mathbf{3}]} \left[\begin{array}{ccc} -1 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 2 & 0 \\ \hline 1 & 2 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{array}\right] = \left[\begin{array}{c} \mathbf{L} \\ \hline \mathbf{E} \end{array}\right].$$

Thus, an associated diagonal matrix is $\mathbf{D} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/2 \end{bmatrix}$, and a basis of eigenvectors is

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}; \begin{pmatrix} 0 \\ -3 \\ 2 \end{pmatrix}; \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \right\}.$$

□

(Final July 12/13) Exercise 2(d)

We can find $\mathbf{A}^{-1}$, from the last section, just computing the inverse of a matrix $\mathbf{S}$ whose columns are linearly independent eigenvectors:

$$\left[\begin{array}{c} \mathbf{S} \\ \hline \mathbf{I} \end{array}\right] = \left[\begin{array}{ccc} 1 & 0 & 1 \\ 0 & -3 & -1 \\ 0 & 2 & 1 \\ \hline 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right] \xrightarrow{[(-1)\overset{\tau}{\mathbf{1}}+\mathbf{3}]} \left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & -3 & -1 \\ 0 & 2 & 1 \\ \hline 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right] \xrightarrow{[\mathbf{2}\overset{\tau}{\rightleftharpoons}\mathbf{3}]} \left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & -1 & -3 \\ 0 & 1 & 2 \\ \hline 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{array}\right] \xrightarrow{\substack{[(-3)\overset{\tau}{\mathbf{2}}+\mathbf{3}] \\ (-1)\mathbf{2}}} \left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & -1 \\ \hline 1 & 1 & 3 \\ 0 & -0 & 1 \\ 0 & -1 & -3 \end{array}\right] \xrightarrow{\substack{[(-1)\overset{\tau}{\mathbf{3}}+\mathbf{2}] \\ (-1)\mathbf{3}}} \left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ \hline 1 & -2 & -3 \\ 0 & -1 & -1 \\ 0 & 2 & 3 \end{array}\right] = \left[\begin{array}{c} \mathbf{I} \\ \hline \mathbf{S}^{-1} \end{array}\right],$$

and then computing the matrix product

$$\mathbf{A}^{-1} = \mathbf{S}\mathbf{D}\mathbf{S}^{-1} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & -3 & -1 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/2 \end{bmatrix} \begin{bmatrix} 1 & -2 & -3 \\ 0 & -1 & -1 \\ 0 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & -1 & -3/2 \\ 0 & 2 & 3/2 \\ 0 & -1 & -1/2 \end{bmatrix}.$$

Or we can directly find the inverse:

$$\left[\begin{array}{ccc} 1 & 2 & 3 \\ 0 & -1 & -3 \\ 0 & 2 & 4 \\ \hline 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right] \xrightarrow{\substack{[(-2)\overset{\tau}{\mathbf{1}}+\mathbf{2}] \\ (-3)\mathbf{1}+\mathbf{3}}} \left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & -1 & -3 \\ 0 & 2 & 4 \\ \hline 1 & -2 & -3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right] \xrightarrow{[(-3)\overset{\tau}{\mathbf{2}}+\mathbf{3}]} \left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 2 & -2 \\ \hline 1 & -2 & 3 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{array}\right] \xrightarrow{[(1)\overset{\tau}{\mathbf{3}}+\mathbf{2}]} \left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \\ \hline 1 & 1 & 3 \\ 0 & -2 & -3 \\ 0 & 1 & 1 \end{array}\right] \xrightarrow{\substack{[(-1)\overset{\tau}{\mathbf{2}}] \\ (-1/2)\mathbf{3}}} \left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ \hline 1 & -1 & -3/2 \\ 0 & 2 & 3/2 \\ 0 & -1 & -1/2 \end{array}\right].$$

$\square$

(Final July 12/13) Exercise 3(a)

First of all, $m = 3$ since $\mathbf{A}\boldsymbol{x} \in \mathbb{R}^3$. In addition, $\mathbf{A}\boldsymbol{x} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ has *one* solution $\implies \mathcal{N}(\mathbf{A}) = \{0\}$, so $r = n$ (where r is the rank of $\mathbf{A}$).

But $\mathbf{A}\boldsymbol{x} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ has no solution $\implies \mathcal{C}(\mathbf{A}) \neq \mathbb{R}^3$, so $r < m = 3$.

There are two possibilities : $\begin{matrix} m = 3 \\ r = n = 1 \end{matrix}$ and $\begin{matrix} m = 3 \\ r = n = 2 \end{matrix}$.

$\square$

(Final July 12/13) Exercise 3(b)

Since $\mathcal{N}(\mathbf{A}) = \{0\}$ (because $\mathbf{A}\boldsymbol{x} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ has 1 solution), there is a unique solution to $\mathbf{A}\boldsymbol{x} = \mathbf{0}$, which is clearly $\boldsymbol{x} = \mathbf{0}$. (Can be either $\boldsymbol{x} = (0)$ or $\boldsymbol{x} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ depending on if $n = 1$ or $n = 2$.)

$\square$

(Final July 12/13) Exercise 3(c)

$\mathbf{A}$ could be $\begin{bmatrix} 0 \\ a \\ 0 \end{bmatrix}$ for $a \neq 0$; or $\begin{bmatrix} 0 & b \\ a & c \\ 0 & d \end{bmatrix}$ or $\begin{bmatrix} b & 0 \\ c & a \\ d & 0 \end{bmatrix}$, for $a \neq 0$ and $b \neq d$, and both columns linearly independent.

$\square$

(Final July 12/13) Short questions set 1(a)

$\det \mathbf{B} = -5$.

$\square$

(Final July 12/13) Short questions set 1(b)

Since the last row is a linear combination of the other two, we know $|\mathbf{C}| = 0$ and therefore an eigenvalue of $\mathbf{C}$ is $\lambda = 0$.

$\square$

(Final July 12/13) Short questions set 2(a)

If, for example, we take as direction vectors of the plane: $\boldsymbol{v} = \boldsymbol{b} - \boldsymbol{a} = (-1, -1, 1)$ and $\boldsymbol{w} = \boldsymbol{c} - \boldsymbol{a} = (0, 0, 1)$, and we choose the point $\boldsymbol{a}$ in the plane; we find the following parametric equation:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \boldsymbol{a} + \alpha\boldsymbol{v} + \beta\boldsymbol{w} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \alpha \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} + \beta \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

$\square$

(Final July 12/13) Short questions set 2(b)

$$\left[\, \boldsymbol{v} \quad \boldsymbol{w} \mid \mathbf{I} \,\right]^{\mathsf{T}} = \left[\begin{array}{ccc} -1 & -1 & -1 \\ 0 & 0 & 1 \\ \hline 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right] \xrightarrow[(-1)\mathbf{1}+\mathbf{3}]{\overset{\boldsymbol{\tau}}{[(-1)\mathbf{1}+\mathbf{2}]}} \left[\begin{array}{ccc} -1 & 0 & 0 \\ 0 & 0 & 1 \\ \hline 1 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right],$$

so the answer is any multiple of $\begin{pmatrix} -1 & 1 & 0 \end{pmatrix}$.

$\square$

(Final July 12/13) Short questions set 3(a)

We just only need the matrix $\begin{bmatrix} 1 & 1 & 1 \\ a & 1 & 1 \\ 1 & c & 1 \end{bmatrix}$ to be rank one. By (leftwards) column eliminaton we get:

$$\begin{bmatrix} 1 & 1 & 1 \\ a & 1 & 1 \\ 1 & c & 1 \end{bmatrix} \xrightarrow[(-1)\mathbf{3}+\mathbf{1}]{\overset{\boldsymbol{\tau}}{[(-1)\mathbf{3}+\mathbf{2}]}} \begin{bmatrix} 0 & 0 & 1 \\ a-1 & 0 & 1 \\ 0 & c-1 & 1 \end{bmatrix}.$$

Hence, $a = c = 1$.

$\square$

(Final July 12/13) Short questions set 3(b)

We just only need to find three pivots after the gaussian elimination. Hence, $a \neq 1$ and $c \neq 1$. We can get the same result by forcing the determinant to be not zero:

$$\begin{vmatrix} 1 & 1 & 1 \\ a & 1 & 1 \\ 1 & c & 1 \end{vmatrix} = ac - c - a + 1 = a(c-1) - c + 1 = (c-1)(a-1) \neq 0,$$

so $a \neq 1, c \neq 1$.

$\square$

(Final July 12/13) Short questions set 4(a)

Since 0 and 2 are the roots of $p(\lambda)$, the matrix $\mathbf{D}$ could be $\begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}$, and then

$$\mathbf{D}^2 - 2\mathbf{D} = \begin{bmatrix} 0 & 0 \\ 0 & 4 \end{bmatrix} - 2\begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix} = \mathbf{0}.$$

If we consider $\mathbf{D} = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$ we get the same result.

$\square$

(Final July 12/13) Short questions set 4(b)

Since $\mathbf{A}$ has no repeated eigenvalues, we can use the diagonalization of the matrix $\mathbf{A} = \mathbf{S}\mathbf{D}\mathbf{S}^{-1}$:

$$\mathbf{A}^2 - 2\mathbf{A} = \mathbf{S}\mathbf{D}^2\mathbf{S}^{-1} - 2\mathbf{S}\mathbf{D}\mathbf{S}^{-1} = \mathbf{S}(\mathbf{D}^2 - 2\mathbf{D})\mathbf{S}^{-1} = \mathbf{S}\mathbf{0}\mathbf{S}^{-1} = \mathbf{0}.$$

$\square$

(Final July 12/13) Short questions set 5(a)

$f(x, y, z) = x^2 + 3y^2 + z^2 - 2xy + 2xz - 2yz.$

$\square$

(Final July 12/13) Short questions set 5(b)

The minors are: $D_1 = 1$, $D_2 = 2$ y $D_3 = 0$. Hence, the quadratic form is positive semi-definite.

$\square$

(Final May 12/13) Exercise 1(a)

By column elimination we get:

$$\left[\begin{array}{c|c} \mathbf{A} & -\boldsymbol{b} \\ \hline \mathbf{I} & \mathbf{0} \end{array}\right] = \left[\begin{array}{ccc|c} 3 & -5 & 1 & 3 \\ \hline 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \xrightarrow[{[1 \rightleftharpoons 3]}]{\tau} \left[\begin{array}{ccc|c} 1 & -5 & 3 & 3 \\ \hline 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{array}\right] \xrightarrow[\substack{[(5)\mathbf{1}+\mathbf{2}] \\ (-3)\mathbf{1}+\mathbf{3} \\ (-3)\mathbf{1}+\mathbf{4}}]{\tau} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 5 & -3 & -3 \end{array}\right] = \left[\begin{array}{c|c} \mathbf{L} & \mathbf{0} \\ \hline \mathbf{E} & \boldsymbol{x}_p \end{array}\right].$$

Hence, the parametric equations are

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \boldsymbol{x}_p + b \cdot \boldsymbol{v} + c \cdot \boldsymbol{w} = \begin{pmatrix} 0 \\ 0 \\ -3 \end{pmatrix} + b\begin{pmatrix} 0 \\ 1 \\ 5 \end{pmatrix} + c\begin{pmatrix} 1 \\ 0 \\ -3 \end{pmatrix}, \quad \text{for all } b, c \in \mathbb{R}.$$

$\square$

(Final May 12/13) Exercise 1(b)

Since the line must be in the plane Π, the vector $\begin{pmatrix} 1 \\ -1 \\ a \end{pmatrix}$ must be a linear combination of $\boldsymbol{v}$ and $\boldsymbol{w}$.

Therefore, solving $\boldsymbol{v}x + \boldsymbol{w}y = \begin{pmatrix} 1 \\ -1 \\ a \end{pmatrix}$ we get:

$$\left[\begin{array}{c|c} [\boldsymbol{v}, \boldsymbol{w}] & -\boldsymbol{b} \\ \hline \mathbf{I} & \mathbf{0} \end{array}\right] = \left[\begin{array}{cc|c} 0 & 1 & -1 \\ 1 & 0 & 1 \\ 5 & -3 & -a \\ \hline 1 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right] \xrightarrow[\substack{[(-1)\mathbf{1}+\mathbf{3}] \\ (1)\mathbf{2}+\mathbf{3}}]{\tau} \left[\begin{array}{cc|c} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 5 & -3 & -a-8 \\ \hline 1 & 0 & -1 \\ 0 & 1 & 1 \end{array}\right]$$

This system is only solvable if $-a-8=0$; hence $\begin{pmatrix}1\\-1\\a\end{pmatrix}$ is only a linear combination of $\boldsymbol{v}$ and $\boldsymbol{w}$ when $\boxed{a=-8}$.

□

(Final May 12/13) Exercise 1(c)

Applying gaussian elimination by columns we get:

$$\left[\begin{array}{ccc}1&-1&-8\\\hline x&y&z\\\hline 0&0&-3\end{array}\right]\xrightarrow[\substack{[(1)\mathbf{1}+\mathbf{2}]\\(8)\mathbf{1}+\mathbf{3}}]{\boldsymbol{\tau}}\left[\begin{array}{ccc}1&0&0\\\hline x&x+y&8x+z\\\hline 0&0&-3\end{array}\right].$$

Thus, the implicit equations of the line are:

$$\begin{cases}x+y&=0\\8x+z&=-3\end{cases}.$$

□

(Final May 12/13) Exercise 2(a)

On the one hand

$$\begin{bmatrix}2&6\\a&b\end{bmatrix}\begin{pmatrix}3\\1\end{pmatrix}=\lambda_1\begin{pmatrix}3\\1\end{pmatrix}\quad\Rightarrow\quad 3\cdot2+6\cdot1=\lambda_1\cdot3\quad\Rightarrow\quad 12=\lambda_1\cdot3\quad\Rightarrow\quad\lambda_1=4;$$

and then $3a+b=4$.

On the other hand

$$\begin{bmatrix}2&6\\a&b\end{bmatrix}\begin{pmatrix}2\\1\end{pmatrix}=\lambda_2\begin{pmatrix}2\\1\end{pmatrix}\quad\Rightarrow\quad 2\cdot2+6\cdot1=\lambda_1\cdot2\quad\Rightarrow\quad 10=\lambda_1\cdot2\quad\Rightarrow\quad\lambda_1=5$$

and then $2a+b=5$.

Hence, $\begin{bmatrix}3&1\\2&1\end{bmatrix}\begin{pmatrix}a\\b\end{pmatrix}=\begin{pmatrix}4\\5\end{pmatrix}$; and therefore

$$\left[\begin{array}{c|c}\mathbf{A}&-\boldsymbol{b}\\\hline\mathbf{I}&\mathbf{0}\end{array}\right]=\left[\begin{array}{cc|c}3&1&-4\\2&1&-5\\\hline1&0&0\\0&1&0\end{array}\right]\xrightarrow{\mathbf{P_{12}}}\left[\begin{array}{cc|c}1&3&-4\\1&2&-5\\\hline0&1&0\\1&0&0\end{array}\right]\xrightarrow[\substack{[(-3)\mathbf{1}+\mathbf{2}]\\(4)\mathbf{1}+\mathbf{3}}]{\boldsymbol{\tau}}\left[\begin{array}{cc|c}1&0&0\\1&-1&-1\\\hline0&1&0\\1&-3&4\end{array}\right]\xrightarrow[{[(-1)\mathbf{2}+\mathbf{3}]}]{\boldsymbol{\tau}}\left[\begin{array}{cc|c}1&0&0\\1&-1&0\\\hline0&1&-1\\1&-3&7\end{array}\right]=\left[\begin{array}{c|c}\mathbf{L}&\mathbf{0}\\\hline\mathbf{E}&\boldsymbol{x}_p\end{array}\right].$$

So, $a=-1$ and $b=7$, and the matrix is $\begin{bmatrix}2&6\\-1&7\end{bmatrix}$.

□

(Final May 12/13) Exercise 2(b)

$$\mathbf{B}=\begin{bmatrix}3&2\\1&1\end{bmatrix}\begin{bmatrix}1&\\&0\end{bmatrix}\begin{bmatrix}3&2\\1&1\end{bmatrix}^{-1}=\begin{bmatrix}3&2\\1&1\end{bmatrix}\begin{bmatrix}1&\\&0\end{bmatrix}\begin{bmatrix}1&-2\\-1&3\end{bmatrix}=\begin{bmatrix}3&-6\\1&-2\end{bmatrix}.$$

So

$$\mathbf{B}^{10}=\begin{bmatrix}3&2\\1&1\end{bmatrix}\begin{bmatrix}1&\\&0\end{bmatrix}^{10}\begin{bmatrix}1&-2\\-1&3\end{bmatrix}=\begin{bmatrix}3&2\\1&1\end{bmatrix}\begin{bmatrix}1^{10}&\\&0^{10}\end{bmatrix}\begin{bmatrix}1&-2\\-1&3\end{bmatrix}=\mathbf{B}.$$

□

(Final May 12/13) Exercise 2(c)

Since two eigenvalues are equal to one ($\lambda=1$), the matrix $\mathbf{C}$ is diagonalizable if $\dim\mathcal{N}(\mathbf{C}-\mathbf{I})=2$. Thus, the two first columns of

$$\mathbf{C}-\mathbf{I}=\begin{bmatrix}0&0&0\\1&-1&0\\1&a&0\end{bmatrix}$$

are linearly dependent, and the matrix $\mathbf{C}$ is diagonalizable, only if $a=-1$.

□

(Final May 12/13) Exercise 3(a)

We are in $\mathbb{R}^3$, which is three-dimensional, so any three linearly independent vectors form a basis as shown in class. Thus, we just need to show that these three vectors are linearly independent, which is equivalent to showing that the 3×3 matrix whose columns are these vectors has full column rank (null space $= \{\mathbf{0}\}$). Proceeding by column elimination:

$$\begin{bmatrix} 1 & 0 & 2 \\ -2 & 2 & 0 \\ 0 & 1 & 0 \end{bmatrix} \xrightarrow[{[(-2)\mathbf{1}+\mathbf{3}]}]{\mathcal{T}} \begin{bmatrix} 1 & 0 & 0 \\ -2 & 2 & 4 \\ 0 & 1 & 0 \end{bmatrix} \xrightarrow[{[(-2)\mathbf{2}+\mathbf{3}]}]{\mathcal{T}} \begin{bmatrix} 1 & 0 & 0 \\ -2 & 2 & 0 \\ 0 & 1 & -2 \end{bmatrix}.$$

Thus, there are three pivots, and hence it has full column rank as desired.

□

(Final May 12/13) Exercise 3(b)

The provided equations multiply $\mathbf{A}$ by three vectors to get three vectors, which by definition of matrix multiplication (recall the column picture) can be combined into a single equation where $\mathbf{A}$ is multiplied by a matrix with three columns to yield a matrix with three columns:

$$\mathbf{A} \begin{bmatrix} 1 & 0 & 2 \\ -2 & 2 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 5 \\ 4 & 0 & 10 \end{bmatrix}.$$

Thus if we take

$$\mathbf{C} = \begin{bmatrix} 2 & 0 & 5 \\ 4 & 0 & 10 \end{bmatrix}$$

and

$$\mathbf{B} = \begin{bmatrix} 1 & 0 & 2 \\ -2 & 2 & 0 \\ 0 & 1 & 0 \end{bmatrix},$$

we have $\mathbf{A} = \mathbf{C}\mathbf{B}^{-1}$. Since $\mathbf{B}$ is precisely the matrix of the basis vectors from part (a), its invertibility follows from above (it is 3×3 and has 3 pivots).

□

(Final May 12/13) Exercise 3(c)

Since $\mathbf{A} = \mathbf{C}\mathbf{B}^{-1}$, we have

$$\mathbf{A}^\top = \left(\mathbf{B}^\top\right)^{-1}\mathbf{C}^\top.$$

(As in class, because $\mathbf{B}$ is invertible, $\mathbf{B}^\top$ is too.)

Since

$$\begin{aligned} \mathbf{A}^\top \boldsymbol{x} = \mathbf{0} \quad \Rightarrow \quad \left(\mathbf{B}^\top\right)^{-1}\mathbf{C}^\top \boldsymbol{x} &= \mathbf{0} \\ \mathbf{B}^\top\left(\mathbf{B}^\top\right)^{-1}\mathbf{C}^\top \boldsymbol{x} &= \mathbf{B}^\top\mathbf{0} \\ \mathbf{C}^\top \boldsymbol{x} &= \mathbf{0} \end{aligned}$$

and

$$\begin{aligned} \mathbf{C}^\top \boldsymbol{x} = \mathbf{0} \quad \Rightarrow \quad \left(\mathbf{B}^\top\right)^{-1}\mathbf{C}^\top \boldsymbol{x} &= \left(\mathbf{B}^\top\right)^{-1}\mathbf{0} \\ \mathbf{A}^\top \boldsymbol{x} &= \mathbf{0}, \end{aligned}$$

then $\mathcal{N}\left(\mathbf{A}^\top\right) = \mathcal{N}\left(\mathbf{C}^\top\right)$. That means we just need to find the null space of $\mathbf{C}^\top$ by elimination:

$$\left[\begin{array}{c} \mathbf{A} \\ \hline \mathbf{I} \end{array}\right] = \left[\begin{array}{cc} 2 & 4 \\ 0 & 0 \\ 5 & 10 \\ \hline 1 & 0 \\ 0 & 1 \end{array}\right] \xrightarrow[{[(-2)\mathbf{1}+\mathbf{2}]}]{\mathcal{T}} \left[\begin{array}{cc} 2 & 0 \\ 0 & 0 \\ 5 & 0 \\ \hline 1 & -2 \\ 0 & 1 \end{array}\right] = \left[\begin{array}{c} \mathbf{L} \\ \hline \mathbf{E} \end{array}\right]$$

in which there is only one free column, so there is one special solution (a basis of the null space)

$$\boldsymbol{s}_1 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$$

or any multiple thereof. (You can also find this special solution by inspection, without elimination.)

□

(Final May 12/13) Exercise 3(d)

Since $\mathbf{A}$ times a 3-vector is a 2-vector, we must have $m = 2$ and $n = 3$. Equivalently, from part (b) we saw that $\mathbf{A}$ was a 2×3 matrix multiplied by a 3×3 matrix, giving a 2×3 matrix. Moreover, from above the dimension of $\mathcal{N}\left(\mathbf{A}^{\mathsf{T}}\right)$ is 1, but this must equal $m - r$, so we obtain $r = 1$.

□

(Final May 12/13) Short questions set 1(a)

$$\begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & 3 & 4 & 5 \\ 0 & 0 & 5 & 6 \\ 1 & 2 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & 3 & 4 & 5 \\ 0 & 0 & 5 & 6 \\ 0 & 0 & -3 & -3 \end{vmatrix} = 1 \cdot 3 \cdot \begin{vmatrix} 5 & 6 \\ -3 & -3 \end{vmatrix} = 3 \cdot (-15 + 18) = 9$$

□

(Final May 12/13) Short questions set 1(b)

Since $\det \mathbf{A} = 9$,

$$x_3 = \frac{1}{9} \begin{vmatrix} 1 & 2 & \mathbf{0} & 4 \\ 0 & 3 & \mathbf{1} & 5 \\ 0 & 0 & \mathbf{0} & 6 \\ 1 & 2 & \mathbf{1} & 1 \end{vmatrix} = \frac{-6}{9} \begin{vmatrix} 1 & 2 & 0 \\ 0 & 3 & 1 \\ 1 & 2 & 1 \end{vmatrix} = -\frac{2}{3}(3 + 2 - 2) = -2$$

□

(Final May 12/13) Short questions set 2(a)

$$\mathbf{A} = \begin{bmatrix} 1 & -1 & 2 \\ -1 & 3 & -4 \\ 2 & -4 & 7 \end{bmatrix}$$

□

(Final May 12/13) Short questions set 2(b)

We can prove that in several alternative ways:

- **Checking the signs of the principal sub-determinants:**

$$1 > 0; \qquad \begin{vmatrix} 1 & -1 \\ -1 & 3 \end{vmatrix} = 2 > 0; \qquad \begin{vmatrix} 1 & -1 & 2 \\ -1 & 3 & -4 \\ 2 & -4 & 7 \end{vmatrix} = 2 > 0.$$

- **Checking the signs of the pivots:** Since the first pivot is 1, and the product of the two first pivots equals the second principal sub-determinant $\begin{vmatrix} 1 & -1 \\ -1 & 3 \end{vmatrix}$, the second pivot is 2. And since the product of the first three pivots equals $\det \mathbf{A}$, the last pivot must be 1. Let's check this is true by column elimination.

$$\left[\begin{array}{rrr} 1 & -1 & 2 \\ -1 & 3 & -4 \\ 2 & -4 & 7 \\ \hline 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right] \xrightarrow[(-2)\mathbf{1}+\mathbf{3}]{\substack{\tau \\ [(1)\mathbf{1}+\mathbf{2}]}} \left[\begin{array}{rrr} 1 & 0 & 0 \\ -1 & 2 & -2 \\ 2 & -2 & 3 \\ \hline 1 & 1 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right] \xrightarrow{\substack{\tau \\ [(1)\mathbf{2}+\mathbf{3}]}} \left[\begin{array}{rrr} 1 & 0 & 0 \\ -1 & 2 & 0 \\ 2 & -2 & 1 \\ \hline 1 & 1 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{array}\right]$$

- **Completing the square for the quadratic form:** from last checking we can see that we can factorize $\mathbf{A}$ in $\mathbf{A} = \dot{\mathbf{L}}\mathbf{D}\dot{\mathbf{U}}$, where $\dot{\mathbf{L}}$ is the transpose of $\dot{\mathbf{U}}$, since $\mathbf{A}$ is symmetric. From the gaussian steps we know that $\dot{\mathbf{U}}$ is

$$\dot{\mathbf{U}} = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 1. \end{bmatrix}$$

And therefore, $f(x, y, z) = 1(x - y + 2z)^2 + 2(y - z)^2 + 1z^2 > 0.$

- **Checking the signs of the eigenvalues of A:** but if we try to find the roots of the characteristic polynomial
$$\begin{vmatrix} 1-\lambda & -1 & 2 \\ -1 & 3-\lambda & -4 \\ 2 & -4 & 7-\lambda \end{vmatrix} = 0$$
we get a 3 degree polynomial, so we need a computer to find the roots. This problem is very common, and therefore, it is better to use any of the other alternative checkings if the order of the matrix is 3 or more.

□

(Final May 12/13) Short questions set 3(a)

Since A and B are orthogonal matrices, then $\mathsf{A}\mathsf{A}^\intercal = \mathsf{I}$ and $\mathsf{B}^{-1} = \mathsf{B}^\intercal$, and therefore $\left(\mathsf{B}^\intercal\right)^{-1} = \mathsf{B}$. Hence, we get
$$\mathsf{A}\mathsf{B}^{-1}(\mathsf{A}\mathsf{B}^{-1})^\intercal = \mathsf{A}\mathsf{B}^{-1}\left(\mathsf{B}^\intercal\right)^{-1}\mathsf{A}^\intercal = \mathsf{A}\mathsf{A}^\intercal = \mathsf{I}.$$

□

(Final May 12/13) Short questions set 3(b)

The order is $m \times m$, so matrix C is square:
$$\mathsf{C} = \underset{m\times n}{\mathsf{B}}\ \underset{n\times n}{(\mathsf{B}^\intercal\mathsf{B})^{-1}}\ \underset{n\times m}{\mathsf{B}^\intercal}\ .$$

And the matrix C^2 is: $\quad \mathsf{C}^2 = \mathsf{B}\underbrace{(\mathsf{B}^\intercal\mathsf{B})^{-1}\mathsf{B}^\intercal\cdot\mathsf{B}}_{\mathsf{I}}(\mathsf{B}^\intercal\mathsf{B})^{-1}\mathsf{B}^\intercal = \mathsf{B}(\mathsf{B}^\intercal\mathsf{B})^{-1}\mathsf{B}^\intercal = \mathsf{C}$

□

(Final May 12/13) Short questions set 3(c)

$||\boldsymbol{v}||^2 = \boldsymbol{v}\cdot\boldsymbol{v} = 4+1+0+16+4 = 25$ so we take $\boldsymbol{u} = \boldsymbol{v}/||\boldsymbol{v}|| = (2/5, -1/5, 0.4/5, -2/5)$.

□

(Final May 12/13) Short questions set 3(d)

The simplest example is
$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

□

(Final May 12/13) Short questions set 4(a)

Both vectors satisfy the equation, so they belong to the subspace of solution. And the set is *linearly independent*. Since the set of solutions is a two dimensional subspace, the set B is a basis.

□

(Final May 12/13) Short questions set 4(b)

False. The matrix $\left(\begin{smallmatrix} a & 0 \\ 0 & a \end{smallmatrix}\right)$ has repeated eigenvalue a, and it is diagonalizable.

□

(Final September 11/12) Exercise 1(a)

Por una parte, $\mathcal{V}_1$ es de dimensión 2 (que es la dimensión del conjunto de soluciones de la ecuación homogénea indicado), es fácil ver que una base de dicho espacio es:
$$\text{una base de } \mathcal{V}_1 \text{ es } \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}; \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} \right\}$$

Por otra parte, $\mathcal{V}_{1/2}$ es de dimensión 1 (que es la dimensión del conjunto de soluciones del sistema de ecuaciones homogéneo indicado), es fácil ver que una base de dicho espacio es:
$$\text{una base de } \mathcal{V}_{1/2} \text{ es } \left\{ \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

Así pues,

$$\mathbf{D} = \begin{bmatrix} 1 & & \\ & 1 & \\ & & \frac{1}{2} \end{bmatrix}, \quad \mathbf{P} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 1 & 1 \end{bmatrix};$$

□

(Final September 11/12) Exercise 1(b)

Primero necesitamos calcular $\mathbf{P}^{-1}$:

$$\left[\mathbf{P}|\mathbf{I}\right] = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{array}\right] = \left[\mathbf{I}|\mathbf{P}^{-1}\right],$$

es decir,

$$\mathbf{P}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 1 & 1 \end{bmatrix}.$$

Por tanto,

$$\mathbf{A} = \mathbf{P}\mathbf{D}\mathbf{P}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ & 1 & \\ & & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -0.5 & 0.5 \end{bmatrix}$$

□

(Final September 11/12) Exercise 1(c)

$\mathbf{M}$ es una matriz de ceros, ya que:

$$\begin{aligned} \mathbf{M} =& 2\mathbf{A}^4 - 7\mathbf{A}^3 + 9\mathbf{A}^2 - 5\mathbf{A} + \mathbf{I} \\ =& 2\mathbf{P}\mathbf{D}^4\mathbf{P}^{-1} - 7\mathbf{P}\mathbf{D}^3\mathbf{P}^{-1} + 9\mathbf{P}\mathbf{D}^2\mathbf{P}^{-1} - 5\mathbf{P}\mathbf{D}\mathbf{P}^{-1} + \mathbf{P}\mathbf{I}\mathbf{P}^{-1} \\ =& \mathbf{P}\left(2\mathbf{D}^4 - 7\mathbf{D}^3 + 9\mathbf{D}^2 - 5\mathbf{D} + \mathbf{I}\right)\mathbf{P}^{-1} \\ =& \mathbf{P}\mathbf{0}\mathbf{P}^{-1} = \mathbf{0}. \end{aligned}$$

□

(Final September 11/12) Exercise 2(a)

All you can say is that rank $\mathbf{A} \leq$ rank $[\mathbf{A}\ \mathbf{B}]$. ($\mathbf{A}$ can have any number r of pivot columns, and these will all be pivot columns for $[\mathbf{A}\ \mathbf{B}]$; but there could be more pivot columns among the columns of $\mathbf{B}$).

□

(Final September 11/12) Exercise 2(b)

Now rank $\mathbf{A}$ = rank $[\mathbf{A}\ \mathbf{A}^2]$. (Every column of $\mathbf{A}^2$ is a linear combination of columns of A. For instance, if we call $\mathbf{A}$'s first column $\mathbf{A}_{|1}$, then $\mathbf{A} \cdot \mathbf{A}_{|1}$ is the first column of $\mathbf{A}^2$. So there are no new pivot columns in the $\mathbf{A}^2$ part of $[\mathbf{A}\ \mathbf{A}^2]$).

□

(Final September 11/12) Exercise 2(c)

The nullspace A has dimension $n - r$, as always. Since $[\mathbf{A}\ \ \mathbf{A}]$ only has r pivot columns — the n columns we added are all duplicates — $[\mathbf{A}\ \mathbf{A}]$ is an m-by-$2n$ matrix of rank r, and its nullspace $N([\mathbf{A}\ \mathbf{A}])$ has dimension $2n - r$.

□

(Final September 11/12) Exercise 3(a)

By gaussian elimination by columns:

$$\left[\begin{array}{c|c}\mathbf{A} & -\boldsymbol{b}\\ \hline \mathbf{I} & \mathbf{0}\end{array}\right] = \left[\begin{array}{ccc|c}1 & 1 & 1 & -3\\ 1 & -1 & 1 & -1\\ 2 & 0 & a & -b\\ \hline 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\end{array}\right] \xrightarrow[(-1)\mathbf{1}+\mathbf{3}]{\underset{[(-1)\mathbf{1}+\mathbf{2}]}{\tau}} \left[\begin{array}{ccc|c}1 & 0 & 0 & -3\\ 1 & -2 & 0 & -1\\ 2 & -2 & a-2 & -b\\ \hline 1 & -1 & -1 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\end{array}\right]$$

$$\xrightarrow{\underset{[(3)\mathbf{1}+\mathbf{4}]}{\tau}} \left[\begin{array}{ccc|c}1 & 0 & 0 & 0\\ 1 & -2 & 0 & 2\\ 2 & -2 & a-2 & -b+6\\ \hline 1 & -1 & -1 & 3\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\end{array}\right] \xrightarrow{\underset{[(1)\mathbf{2}+\mathbf{4}]}{\tau}} \left[\begin{array}{ccc|c}1 & 0 & 0 & 0\\ 1 & -2 & 0 & 0\\ 2 & -2 & a-2 & -b+4\\ \hline 1 & -1 & -1 & 2\\ 0 & 1 & 0 & 1\\ 0 & 0 & 1 & 0\end{array}\right]$$

No solution if $a = 2$ and $b \neq 4$; since it is not possible transforming the fourth column into a zero vector if the third one is also zero.

□

(Final September 11/12) Exercise 3(b)

The number of zero columns (in the coeficient part of the matrix) is the dimension of the space of solutions

If $a \neq 2$ then there is no zero columns in the coefficients matrix part, and hence the dimension of $\mathcal{N}\left(\mathbf{A}\right)$ is 0. When $a = 2$ there is only one zero column, and the dimension of $\mathcal{N}\left(\mathbf{A}\right)$ is 1. In this case the set of solution can't be a plane.

□

(Final September 11/12) Exercise 3(c)

When $a = 2$ the system has solution only if $b = 4$ (véase la respuesta al primer apartado). Hence, the dimension of $\mathcal{N}\left(\mathbf{A}\right)$ is one, and the set of solutions is a line.

Since the first and the last columns of $\mathbf{A}$ are equal, we can chose as unique free variable ($\dim \mathcal{N}\left(\mathbf{A}\right) = 1$) just only one of then: the first or the last variable.

When $a = 2$ la tercera columna de la parte de la matriz de coeficientes es una columna de ceros.

$$\left[\begin{array}{ccc|c}1 & 0 & 0 & 0\\ 1 & -2 & 0 & 0\\ 2 & -2 & 0 & -4+4\\ \hline 1 & -1 & -1 & 2\\ 0 & 1 & 0 & 1\\ 0 & 0 & 1 & 0\end{array}\right]$$

el vector que aparece debajo, es una base del espacio solución del *sistema homogéneo*.

$$\text{basis} = \left\{ \begin{pmatrix}-1\\ 0\\ 1\end{pmatrix} \right\}.$$

Pero el sistema que estamos resolviendo *no es homogéneo* ($\boldsymbol{b} \neq \mathbf{0}$), y por tanto este conjunto de soluciones no es una recta que pasa por el origen (no es un espacio vectorial); así que no podemos encontrar una base para el conjunto de soluciones de este sistema con $\boldsymbol{b} \neq \mathbf{0}$.

□

(Final September 11/12) Exercise 3(d)

En este caso, tras la eliminación gausiana, obtenemos

$$\left[\begin{array}{ccc|c}1 & 0 & 0 & 0\\ 1 & -2 & 0 & 0\\ 2 & -2 & 1 & 0\\ \hline 1 & -1 & -1 & 2\\ 0 & 1 & 0 & 1\\ 0 & 0 & 1 & 0\end{array}\right]$$

Este sistema es compatible (hemos logrado hacer una columna de ceros en la parte del vector del lado derecho, $\boldsymbol{b}$) y determinado (no hay columnas de ceros en la parte de la matriz de coeficientes).

El vector solución aparece debajo del vector de ceros de la derecha.

$$\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix};$$

es decir, $x = 2$, $y = 1$, y $z = 0$.

Este vector pertenece al conjunto de soluciones del apartado anterior (¡nótese como ya aparecía este vector en el apartado anterior!).

□

(Final September 11/12) Short questions set 1(a)

We first have to find a vector in the direction of the line. We let

$$\boldsymbol{v} = \begin{pmatrix} -1 \\ 2 \end{pmatrix} - \begin{pmatrix} 0 \\ 3 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \end{pmatrix}.$$

A parametric representation of the line is therefore

$$\boldsymbol{x} = \boldsymbol{x}_P + a\boldsymbol{v} \quad \Rightarrow \quad \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \end{pmatrix} + a\begin{pmatrix} -1 \\ -1 \end{pmatrix} \quad \text{or} \quad \begin{cases} x_1 = -a \\ x_2 = 3 - a \end{cases}.$$

□

(Final September 11/12) Short questions set 1(b)

$$\left[\begin{array}{c|cc} -1 & 1 & 0 \\ -1 & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{c|cc} -1 & 1 & 0 \\ 0 & -1 & 1 \end{array}\right]$$

hence

$$\begin{pmatrix} -1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 3 \end{pmatrix} + a\begin{pmatrix} -1 & 1 \end{pmatrix}\begin{pmatrix} -1 \\ -1 \end{pmatrix} \Rightarrow \left\{ -x + y = 3 \right. .$$

□

(Final September 11/12) Short questions set 2.

Puesto que el determinante es negativo independientemente del valor de b:

$$\begin{vmatrix} 2 & b & 3 \\ b & 2 & b \\ 3 & b & 4 \end{vmatrix} = 16 + 6b^2 - 18 - 6b^2 = -2 \quad < 0;$$

esta matriz nunca puede tener sus tres autovalores positivos.

□

(Final September 11/12) Short questions set 3(a)

$\det \mathbf{A}^\intercal \mathbf{A} = \det \underset{3\times 3}{\mathbf{I}} = 1$

□

(Final September 11/12) Short questions set 3(b)

$\mathbf{AA}^\intercal$ es de orden 5 por 5 pero su rango es sólo 3 (ya que es una matriz diagonal con tres unos y dos ceros en la diagonal principal); por tanto $\mathbf{AA}^\intercal$ es singular y $\det \mathbf{AA}^\intercal = 0$.

Otro razonamiento para ver que la matriz es singular es:

1. The 3 by 5 matrix $\mathbf{A}^\intercal$ has 3 linearly independent (orthonormal!) rows and a nontrivial nullspace of dimension $5 - r = 5 - 3 = 2$.
2. Then $\mathbf{AA}^\intercal$ must have dependent columns because $\mathbf{A}(\mathbf{A}^\intercal \boldsymbol{y}) = 0$ for any nonzero vector $\boldsymbol{y}$ in the nullspace of $\mathbf{A}^\intercal$.
3. Hence, $\det \mathbf{AA}^\intercal = 0$.

□

(Final September 11/12) Short questions set 3(c)

$$\det \mathbf{A} \underbrace{(\mathbf{A}^\intercal \mathbf{A})^{-1}}_{\mathbf{I}} \mathbf{A}^\intercal = \det \mathbf{AA}^\intercal = 0.$$

□

(Final September 11/12) Short questions set 4.

$\det(\mathbf{A}) = 5x^2 - 6x + 0 - 9x + 10x - 0 = 5x^2 - 5x = 5x(x-1) = 0$. Therefore, $x = 0, x = 1$.

□

(Final September 11/12) Short questions set 5.

$$\mathbf{A}\boldsymbol{v} = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 1 & 3 \\ 4 & 2 & 3 & 1 \\ 3 & 1 & 4 & 2 \end{bmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 10 \\ 10 \\ 10 \\ 10 \end{pmatrix} = 10\boldsymbol{v}.$$

□

(Final September 11/12) Short questions set 6(a)

$$(\mathbf{A}^\intercal \mathbf{B}^\intercal)^{-1} = \Big((\mathbf{A}\mathbf{B})^\intercal\Big)^{-1} = \Big((\mathbf{A}\mathbf{B})^{-1}\Big)^\intercal = (\mathbf{A}^{-1}\mathbf{B}^{-1})^\intercal.$$

Es verdadero.

□

(Final September 11/12) Short questions set 6(b)

Si $\mathbf{A}$ y $\mathbf{B}$ son además ortonormales entonces $\mathbf{A}\mathbf{A}^\intercal = \mathbf{I}$ y $\mathbf{B}\mathbf{B}^\intercal = \mathbf{I}$; así pues,

$$\mathbf{A}\mathbf{B}(\mathbf{A}\mathbf{B})^\intercal = \mathbf{A}\mathbf{B}\mathbf{B}^\intercal\mathbf{A}^\intercal = \mathbf{A}\mathbf{I}\mathbf{A}^\intercal = \mathbf{A}\mathbf{A}^\intercal = \mathbf{I}.$$

Es verdadero.

□

(Final June 11/12) Exercise 1(a)

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 2 \\ 0 & 1 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The rank is 3; therefore, the columns are linearly dependent.

□

(Final June 11/12) Exercise 1(b)

$$\mathcal{V} = \left\{ \boldsymbol{x} \in \mathbb{R}^4 \quad \text{such as} \quad \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = a \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + b \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} + c \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right\}$$

This is a parametric representation of the vector space $\mathcal{V}$. For an implicit representation, we must multiply by a vector orthogonal to the three first columns. The set of solutions of the resulting system is also $\mathcal{V}$:

$$\begin{pmatrix} 0 & -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = 0 \quad \Rightarrow \quad -x_2 + x_4 = 0.$$

The dimension of this subspace of $\mathbb{R}^4$ is three. Nothing changes if we include also the fourth column, since it is a linear combination of the first three columns of $\mathbf{A}$.

□

(Final June 11/12) Exercise 1(c)

From the first part of this exercice it is easy to see that $\mathcal{N}\left(\mathbf{A}\right)$ is:

$$\left\{ \boldsymbol{x} \in \mathbb{R}^4 \quad \text{such as} \quad \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = a \begin{pmatrix} -1 \\ -1 \\ -1 \\ 1 \end{pmatrix} \quad \text{for all} \quad a \in \mathbb{R} \right\}.$$

There is one exogenous (or free) variable. Any variable can be choosen in this case. The dimension of $\mathcal{N}\left(\mathbf{A}\right)$ is one.

□

(Final June 11/12) Exercise 2(a)

Both are always always diagonalizable, since $\mathbf{A}$ is symmetric (and so it is $\mathbf{A}^{-1}$)[2] (but $\mathbf{A}^{-1}$ does not exist if $a = 0$).

□

(Final June 11/12) Exercise 2(b)

$$\det(\mathbf{A} - \lambda\mathbf{I}) = \begin{vmatrix} 1-\lambda & 0 & 0 \\ 0 & -\lambda & 1 \\ 0 & 1 & -\lambda \end{vmatrix} = \lambda^2(1-\lambda) - (1-\lambda) = 0 \quad \longrightarrow \quad \begin{cases} \lambda = 1 \\ \lambda = 1 \\ \lambda = -1 \end{cases}$$

For $\lambda = 1$

$$\mathbf{A} - \mathbf{I} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \end{bmatrix}; \qquad \text{Orthonormal basis:} \quad \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}; \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} \right\}$$

For $\lambda = -1$

$$\mathbf{A} - \mathbf{I} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}; \qquad \text{Orthonormal basis:} \quad \left\{ \begin{pmatrix} 0 \\ \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} \right\}$$

Hence

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 1 & & \\ & 1 & \\ & & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

□

(Final June 11/12) Exercise 2(b)

Same answer since if $\mathbf{A} = \mathbf{S}\mathbf{D}\mathbf{S}^{-1}$ then $\mathbf{A}^{-1} = \mathbf{S}\mathbf{D}^{-1}\mathbf{S}^{-1}$; same matrix $\mathbf{S}$.

In addition, in this very particular case $\lambda = \lambda^{-1}$ (also the same eigenvalues!). Please note that $\mathbf{A}\mathbf{A} = \mathbf{I}$, so, in this case $\boxed{\mathbf{A} = \mathbf{A}^{-1}}$ (since $\mathbf{A}$ is a symmetric permutation matrix when $a = 1$ and $b = 0$).

□

(Final June 11/12) Exercise 2(b)

$$\mathbf{A}\boldsymbol{u} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{pmatrix} 0 \\ 2 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \\ 2 \end{pmatrix} = \boldsymbol{u}; \qquad \text{hence} \qquad \mathbf{A}^{10}\boldsymbol{u} = \boldsymbol{u}.$$

□

(Final June 11/12) Exercise 3(a)

A set of two vectors is linearly independent if and only if one vector is a multiple of the other. This is not the case, hence, they are linearly independent. We can also check that the matrix: $\begin{bmatrix} \boldsymbol{x}_1, & \boldsymbol{x}_2 \end{bmatrix}$ has rank 2.

On the other hand

$$\begin{pmatrix} -2 & -1 & 3 & 4 \end{pmatrix} \begin{pmatrix} -8 \\ 2 \\ -2 \\ 1 \end{pmatrix} = 12,$$

therefore this vectors are an orthogonal set.

□

(Final June 11/12) Exercise 3(b)

No, since the first and the last vector are equal.

□

(Final June 11/12) Exercise 3(c)

The $\boldsymbol{x}_2$ and $\boldsymbol{x}_3$ vectors are not solutions to the system. Therefore, the set is not a basis for the 3-dimensional subspace of solutions.

□

(Final June 11/12) Exercise 3(d)

[2]If $\mathbf{A} = \mathbf{A}^\intercal$ and $\mathbf{B} = \mathbf{A}^{-1}$ then $\mathbf{A}\mathbf{B} = \mathbf{I} \Rightarrow \mathbf{B}^\intercal\mathbf{A}^\intercal = \mathbf{B}^\intercal\mathbf{A} = \mathbf{I}$; therefore $\mathbf{B}^\intercal = \mathbf{A}^{-1} = \mathbf{B}$, that is, $\mathbf{A}^{-1}$ is also symmetric.

$$\begin{bmatrix} 1 & 0 & -1 & q \\ 4 & 2 & 12 & 3 \\ 6 & 2 & 10 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1 & q \\ 0 & 2 & 16 & 3-4q \\ 0 & 2 & 16 & 1-6q \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1 & q \\ 0 & 2 & 16 & 3-4q \\ 0 & 0 & 0 & -2-2q \end{bmatrix}.$$

So these vector do not span $\mathbb{R}^3$ if $q = -1$ (rank 2).

□

(Final June 11/12) Short questions set 1(a)

$$\det \mathbf{A} = \begin{vmatrix} 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 3 \\ 1 & 3 & 1 & 7 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 0 \\ 1 & 2 & 3 \\ 1 & 3 & 7 \end{vmatrix} = 1; \qquad \Rightarrow \qquad \det \mathbf{A}^{-1} = 1.$$

□

(Final June 11/12) Short questions set 1(b)

The $(1,2)$ entry of $\mathbf{A}^{-1}$ is $\frac{\text{cof}(\mathbf{A})_{2,1}}{\det \mathbf{A}}$; hence

$$\frac{\text{cof}(\mathbf{A})_{2,1}}{\det \mathbf{A}} = \frac{-\begin{vmatrix} 0 & 1 & 0 \\ 2 & 1 & 3 \\ 3 & 1 & 7 \end{vmatrix}}{1} = 5.$$

□

(Final June 11/12) Short questions set 2(a)

Using the definition of eigenvectors and eigenvalues ($\mathbf{A}\boldsymbol{v} = \lambda \boldsymbol{v}$), it is easy to see that the eigenvalues are $\lambda_1 = 6$, $\lambda_2 = 3$, $\lambda_3 = 3$; and $\boldsymbol{x}_1 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$, $\boldsymbol{x}_2 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$ and $\boldsymbol{x}_3 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$ are eigenvectors corresponding to those eigenvalues.

□

(Final June 11/12) Short questions set 2(b)

The matrix $\mathbf{A}$ is diagonalizable since $\boldsymbol{x}_2$ and $\boldsymbol{x}_3$ are linearly independent.
The matrix $\mathbf{A}$ is invertible since all its eigenvalues are different from cero.

□

(Final June 11/12) Short questions set 2(c)

$\det \mathbf{A} = \lambda_1 \cdot \lambda_2 \cdot \lambda_3 = 54$
$\text{tr}\left(\mathbf{A}\right) = \lambda_1 + \lambda_2 + \lambda_3 = 12$

□

(Final June 11/12) Short questions set 2(d)

Yes it is; the three eigenvector $\boldsymbol{x}_1$, $\boldsymbol{x}_2$ and $\boldsymbol{x}_3$ is an orthogonal set.

□

(Final June 11/12) Short questions set 3.

Since the coefficient associated to x_1^2 is positive $(+1)$, this quadratic form cannot be negative definite.

□

(Final June 11/12) Short questions set 4(a)

True, since $\det \mathbf{A} =$ *the product of its eigenvalues*, and therefore $\det \mathbf{A} = 0$.

□

(Final June 11/12) Short questions set 4(b)

True. If $\lambda = -3$ is an eigenvalue of $\mathbf{A}$, then $(\mathbf{A} + 3\mathbf{I})$ must be singular ($(\det(\mathbf{A} + 3\mathbf{I}) = 0)$.

□

(Final June 11/12) Short questions set 4(c)

False. If $\lambda = 0$ is an eigenvalue of $\mathbf{A}$, then $\mathbf{A}$ is singular; and therefore $\mathbf{A}\boldsymbol{x} = \mathbf{0}$ has infinite solutions.

□

(Final September 10/11) Exercise 1(a)

$$\det \mathbf{A} = \begin{vmatrix} 1 & 1 & 0 & 1 \\ 1 & a & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 & 1 \\ 1 & a & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{vmatrix} = - \begin{vmatrix} 1 & 0 & 0 & 0 \\ 1 & a & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{vmatrix} = -a;$$

therefore $|\mathbf{A}|$ is no zero if and only if $a \neq 0$.

□

(Final September 10/11) Exercise 1(b)

No, since:

$$|1| = 1; \quad \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} = 0; \quad \begin{vmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 0; \quad \begin{vmatrix} 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{vmatrix} = -1,$$

when all subdeterminants should be possitive. It follows that the matrix is not definite.

□

(Final September 10/11) Exercise 1(c)

$$\left(\begin{array}{cccc|cccc} 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right) \to \left(\begin{array}{cccc|cccc} 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & -1 & -1 & 0 & 0 & 1 \end{array}\right) \to \left(\begin{array}{cccc|cccc} 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -2 & -2 & 1 & 0 & 1 \end{array}\right) \to$$

$$\left(\begin{array}{cccc|cccc} 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & -1/2 & 0 & -1/2 \end{array}\right) \to \left(\begin{array}{cccc|cccc} 1 & 1 & 0 & 0 & 0 & 1/2 & 0 & 1/2 \\ 0 & 1 & 0 & 0 & 0 & 1/2 & 0 & -1/2 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & -1/2 & 0 & -1/2 \end{array}\right) \to$$

$$\left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1/2 & 0 & -1/2 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & -1/2 & 0 & -1/2 \end{array}\right)$$

Hence,

$$\mathbf{A}^{-1} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1/2 & 0 & -1/2 \\ 0 & 0 & 1 & 0 \\ 1 & -1/2 & 0 & -1/2 \end{pmatrix}$$

□

(Final September 10/11) Exercise 1(d)

From the steps given in the gaussian elimination when solving the first part of the exercise, it's easy to check that $\operatorname{rg}(\mathbf{A}) = 3$ when $a = 0$; and therefore, there are three pivot variables. Hence, only one variable can be chosen as a free variable.

When $a = 0$ the second and third columns are equal (and hence dependent); it follows that we can take as free variable either the second or the third one.

□

(Final September 10/11) Exercise 2(a)

Since the matrix is triangular, the elements on its main diagonal ($\lambda = 4$ and $\lambda = 2$) are the eigenvalues (both with algebraic multiplicity two):

$$\operatorname{rg}(\mathbf{A} - 4\mathbf{I}) = \operatorname{rg}\left(\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 1 & 0 & 0 & -2 \end{bmatrix} \right) = 2; \quad \operatorname{rg}(\mathbf{A} - 2\mathbf{I}) = \operatorname{rg}\left(\begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \right) = 2.$$

Por tanto ya sabemos que $\mathbf{A}$ es diagonalizable.

Observando la matriz $\mathbf{A}$-4$\mathbf{I}$, es fácil ver que dos autovalores asociados a $\lambda = 4$ son

$$\begin{pmatrix} 2 \\ 0 \\ 0 \\ 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix};$$

y observando la matriz $\mathbf{A}$-4$\mathbf{I}$, que dos autovalores asociados a $\lambda = 2$ son

$$\begin{pmatrix}0\\0\\1\\0\end{pmatrix} \quad \text{and} \quad \begin{pmatrix}0\\0\\0\\1\end{pmatrix}$$

Así pues,

$$\mathbf{D} = \begin{bmatrix}4&&&\\&4&&\\&&2&\\&&&2\end{bmatrix}; \qquad \mathbf{S} = \begin{bmatrix}2&0&0&0\\0&1&0&0\\0&0&1&0\\1&0&0&1\end{bmatrix}.$$

□

(Final September 10/11) Exercise 2(b)

Puesto que hemos visto que $\boldsymbol{v}$ es un autovector de $\mathbf{A}$ asociado al autovalor 2, sabemos que $\mathbf{A}\boldsymbol{v} = 2\boldsymbol{v}$, y por tanto:

$$\begin{aligned}\mathbf{A}^6\boldsymbol{v} =&\mathbf{A}\cdot\mathbf{A}\cdot\mathbf{A}\cdot\mathbf{A}\cdot\mathbf{A}\cdot\mathbf{A}\cdot\boldsymbol{v}\\ =&\mathbf{A}\cdot\mathbf{A}\cdot\mathbf{A}\cdot\mathbf{A}\cdot\mathbf{A}\cdot 2\boldsymbol{v}\\ =&\mathbf{A}\cdot\mathbf{A}\cdot\mathbf{A}\cdot\mathbf{A}\cdot 4\boldsymbol{v}\\ =&\mathbf{A}\cdot\mathbf{A}\cdot\mathbf{A}\cdot 8\boldsymbol{v}\\ =&\mathbf{A}\cdot\mathbf{A}\cdot 16\boldsymbol{v}\\ =&\mathbf{A}\cdot 32\boldsymbol{v} = \lambda^6\boldsymbol{v}\\ =&2^6\boldsymbol{v} = 64\boldsymbol{v} = \begin{pmatrix}0&0&0&64\end{pmatrix}.\end{aligned}$$

□

(Final September 10/11) Exercise 2(c)

Puesto que ningún autovalor es cero, la matriz es de rango completo, es decir, invertible.

□

(Final September 10/11) Exercise 2(d)

Puesto que $\mathbf{A} = \mathbf{SDS}^{-1}$, entonces

$$\mathbf{A}^{-1} = \left(\mathbf{SDS}^{-1}\right)^{-1} = \left(\mathbf{DS}^{-1}\right)^{-1}\mathbf{S}^{-1} = \left(\mathbf{S}^{-1}\right)^{-1}\mathbf{D}^{-1}\mathbf{S}^{-1} = \mathbf{SD}^{-1}\mathbf{S}^{-1};$$

es decir, los autovectores $\mathbf{S}$ son los mismos, pero los autovalores $\mathbf{D}^{-1}$, son los inversos de los autovalores de la matriz $\mathbf{A}$.

□

(Final September 10/11) Exercise 3(a)

$$[\mathbf{A}|\boldsymbol{b}] = \left[\begin{array}{ccccc|c}1&2&0&1&1&1\\0&0&2&3&1&0\\0&0&1&4&2&1\\0&0&0&1&1&2\end{array}\right] \to \left[\begin{array}{ccccc|c}1&2&0&1&1&1\\0&0&1&4&2&1\\0&0&2&3&1&0\\0&0&0&1&1&2\end{array}\right] \to \left[\begin{array}{ccccc|c}1&2&0&1&1&1\\0&0&1&4&2&1\\0&0&0&-5&-3&-2\\0&0&0&1&1&2\end{array}\right]$$

$$\to \left[\begin{array}{ccccc|c}1&2&0&1&1&1\\0&0&1&4&2&1\\0&0&0&1&1&2\\0&0&0&-5&-3&-2\end{array}\right] \to \left[\begin{array}{ccccc|c}1&2&0&1&1&1\\0&0&1&4&2&1\\0&0&0&1&1&2\\0&0&0&0&2&8\end{array}\right] \to \left[\begin{array}{ccccc|c}1&2&0&1&1&1\\0&0&1&4&2&1\\0&0&0&1&1&2\\0&0&0&0&1&4\end{array}\right]$$

$$\to \left[\begin{array}{ccccc|c}1&2&0&1&0&-3\\0&0&1&4&0&-7\\0&0&0&1&0&-2\\0&0&0&0&1&4\end{array}\right] \to \left[\begin{array}{ccccc|c}1&2&0&0&0&-1\\0&0&1&0&0&1\\0&0&0&1&0&-2\\0&0&0&0&1&4\end{array}\right] = [\mathbf{R}|\boldsymbol{c}].$$

Por lo que la solución completa es:

$$\textit{todo vector } \boldsymbol{x} \textit{ de la forma:} \quad \boldsymbol{x} = a\begin{pmatrix}-2\\1\\0\\0\\0\end{pmatrix} + \begin{pmatrix}-1\\0\\1\\-2\\4\end{pmatrix}; \quad \textit{para cualquier } a \in \mathbb{R}.$$

Es decir, el conjunto de vectores $\boldsymbol{x}$ de la forma:

$$\begin{pmatrix}x_1\\x_2\\x_3\\x_4\\x_5\end{pmatrix}=\begin{pmatrix}-1-a\\a\\1\\-2\\4\end{pmatrix}\ \textit{para cualquier } a\in\mathbb{R};\quad \text{o}\quad \begin{pmatrix}x_1\\x_2\\x_3\\x_4\\x_5\end{pmatrix}=\begin{pmatrix}-1-x_2\\x_2\\1\\-2\\4\end{pmatrix}\ \textit{para cualquier } x_2\in\mathbb{R}.$$

□

(Final September 10/11) Exercise 3(b)

Puesto que el sistema tiene cinco incognitas, el vector solución tiene cinco elementos (un valor para cada incognita). Así pues, el conjunto de soluciones es un subconjunto de $\mathbb{R}^5$; Y en este caso, dicho conjunto es una recta, ya que sólo una de las columnas de $\mathbf{A}$ es dependiente de las demás (sólo hay una variable libre o exógena; sólo un parámetro o grado de libertad en la solución). Así pues, un vector director es cualquier múltiplo (excepto el vector nulo $\mathbf{0}$) de la solución al sistema homogéneo que hemos encontrado: $\boldsymbol{x}_a = \begin{pmatrix}-2 & 1 & 0 & 0 & 0\end{pmatrix}$. Y uno de los puntos por donde pasa la recta es la solución particular que obtuvimos al resolver el sistema: $\boldsymbol{x}_p = \begin{pmatrix}-1 & 0 & 1 & -2 & 4\end{pmatrix}$

□

(Final September 10/11) Exercise 3(c)

Primero un razonamiento largo...

Está claro que el vector director $\boldsymbol{x}_a$ (la solución al sistema homogeneo) cumple la siguiente relación:

$$\mathbf{A}\boldsymbol{x}_a=\mathbf{0};$$

lo cual significa que los vectores fila de $\mathbf{A}$ son perpendiculares a $\boldsymbol{x}_a$. Así que, al menos, las filas de $\mathbf{A}$ son perpendiculares a $\boldsymbol{x}_a$.

Pero... ¿hay más vectores perpendiculares? Veamos si cualquier combinación lineal de las filas de $\mathbf{A}$ es un nuevo vector perpendicular a $\boldsymbol{x}_a$.

Sea $\boldsymbol{z}$ un vector de $\mathbb{R}^4$, entonces $\boldsymbol{z}\mathbf{A}$ es un nuevo vector de $\mathbb{R}^4$ generado como combinación lineal de las filas de $\mathbf{A}$ (donde los elementos z_i de $\boldsymbol{z}$ son los coeficientes de dicha combinación). Por tanto, para todo $\boldsymbol{z}\in\mathbb{R}^4$, el producto $\boldsymbol{z}\mathbf{A}$ es una combinación lineal de las filas de $\mathbf{A}$. Comprobar que las combinaciones $\boldsymbol{z}\mathbf{A}$ son siempre perpendiculares al vector director $\boldsymbol{x}_a$ es muy sencillo, ya que si $\mathbf{A}\boldsymbol{x}_a=\mathbf{0}$ entonces, para el producto de cualquier combinación lineal de filas $\boldsymbol{z}\mathbf{A}$ con el vector director $\boldsymbol{x}_a$ siempre resultará que

$$\boldsymbol{z}\mathbf{A}\boldsymbol{x}_a=\boldsymbol{z}\cdot\mathbf{0}=0.$$

¿Hemos encontrado todos los vectores perpendiculares a $\boldsymbol{x}_a$? o ¿hay algún vector en $\mathbb{R}^5$ que sea perpendicular a $\boldsymbol{x}_a$, pero que no sea combinación de las filas de $\mathbf{A}$?

Para contestar a estas dos preguntas primero vamos a comprobar que el conjunto de vectores perpendiculares a $\boldsymbol{x}_a$ son un espacio vectorial; es decir, que dicho conjunto es cerrado para la suma y el producto por un escalar (o de manera más abreviada, que es cerrado para las combinaciones lineales). Veámoslo:

Sean $\boldsymbol{y}$ y $\boldsymbol{z}$ dos vectores perpendiculares a $\boldsymbol{x}_a$, es decir, dos vectores tales que $\boldsymbol{y}\cdot\boldsymbol{x}_a=0$ y que $\boldsymbol{z}\cdot\boldsymbol{x}_a=0$; y sea $\mathbf{B}$ la matriz cuyas filas son $\boldsymbol{y}$ y $\boldsymbol{z}$.

De nuevo, una combinación de dichas filas es el producto

$$\boldsymbol{c}\mathbf{B}=\mathbf{B}^{\intercal}\boldsymbol{c}=\begin{bmatrix}\boldsymbol{y} & \boldsymbol{z}\end{bmatrix}\begin{bmatrix}c_1\\c_2\end{bmatrix}$$

para cualquier vector $\boldsymbol{c}$ de $\mathbb{R}^2$. Ahora vamos a comprobar que todas las posibles combinaciones de dos vectores perpendiculares al vector $\boldsymbol{x}_a$ son también perpendiculares a dicho vector director (que dicho conjunto es cerrado).

$$\boldsymbol{c}\mathbf{B}\boldsymbol{x}_a=\boldsymbol{x}_a\mathbf{B}^{\intercal}\boldsymbol{c}=\boldsymbol{x}_a\begin{bmatrix}\boldsymbol{y} & \boldsymbol{z}\end{bmatrix}\boldsymbol{c}=\begin{bmatrix}\boldsymbol{x}_a\cdot\boldsymbol{y} & \boldsymbol{x}_a\cdot\boldsymbol{z}\end{bmatrix}\boldsymbol{c}=\begin{bmatrix}0 & 0\end{bmatrix}\boldsymbol{c}=0,\qquad \text{para todo } \boldsymbol{c}\in\mathbb{R}^2.$$

Así pues, el conjunto de vectores perpendiculares a $\boldsymbol{x}_a$ es un subespacio vectorial. ¿De que dimensión?

Tanto los vectores fila de $\mathbf{A}$ como el vector director $\boldsymbol{x}_a$ tienen cinco componentes, es decir, pertenencen a $\mathbb{R}^5$, que es un espacio vectorial de dimensión cinco. Puesto que $\mathbf{A}$ tiene rango 4, sus cuatro filas son linealmente independientes, y constituyen una base del espacio generado por las filas de $\mathbf{A}$; que es un subespacio de dimensión 4 y que llamaremos espacio fila de $\mathbf{A}$. Por supuesto, la recta generada por el único vector director $\boldsymbol{x}_a$ es de dimensión 1.

Así pues, la unión del espacio fila de $\mathbf{A}$ (dimensión 4) junto con los vectores de la *recta perpendicular* (dimensión 1) tiene necesáriamente dimensión 5, es decir, la unión de los dos subespacios es todo el espacio

$\mathbb{R}^5$. Eso significa que cualquier vector de $\mathbb{R}^5$, o está en la recta generada por $\boldsymbol{x}_a$, o está en el espacio fila de $\mathbf{A}$; y por lo tanto, todo vector perpendicular a $\boldsymbol{x}_a$ necesariamente pertenece al espacio filas de $\mathbf{A}$.

Ya podemos contestar a todas las preguntas: el conjunto de vectores perpendiculares es el espacio fila de $\mathbf{A}$*, que es de dimensión 4; y como* $\mathbf{A}$ *tiene rango 4, sus cuatro filas son linealmente independientes, así que constituyen una base del subespacio de vectores perpendiculares a* $\boldsymbol{x}_a$*.*

Y ahora un razonamiento más corto... en el que sólo es necesario resolver el sistema de ecuaciones "apropiado"... del que en este caso particular, y dado lo que ya sabemos del primer apartado... ya conocemos su solución...

Antes hemos recordado que en cualquier sistema homogeneo $\mathbf{A}\boldsymbol{x} = \mathbf{0}$, los vectores solución $\boldsymbol{x}$ son los vectores ortogonales a las filas de la matriz de coeficientes $\mathbf{A}$ ... pero entonces... para contestar a este apartado ¡basta con poner como coeficientes del sistema homogéneo los elementos de vector director $\boldsymbol{x}_a = \begin{pmatrix}-2 & 1 & 0 & 0 & 0\end{pmatrix}$, y resolver! Es decir, la pregunta se puede contestar solucionando el sistema

$$\begin{bmatrix}-2 & 1 & 0 & 0 & 0\end{bmatrix}\begin{pmatrix}x_1\\x_2\\x_3\\x_4\\x_5\end{pmatrix} = 0.$$

El conjunto de vectores solución a este sistema es el conjunto de vectores ortogonales pedidos en el enunciado, y puesto que la matriz de coeficientes tiene rango uno, y hay cinco incógnitas, *la dimensión del conjunto solución es cuatro.*

Probar que dicho conjunto es un subespacio vectorial es fácil: si $\boldsymbol{y}$ y $\boldsymbol{z}$ son soluciones al sistema $\mathbf{B}\boldsymbol{x} = \mathbf{0}$, entonces la combinación lineal $a\boldsymbol{y} + b\boldsymbol{z}$ también es solucion puesto que

$$\mathbf{B}(a\boldsymbol{y} + b\boldsymbol{z}) = a\mathbf{B}\boldsymbol{y} + b\mathbf{B}\boldsymbol{z} = a\mathbf{0} + b\mathbf{0} = \mathbf{0}.$$

Observando el primer apartado de este problema podemos ver que *las cuatro filas del sistema de ecuaciones original* $\mathbf{A}\boldsymbol{x}_a = \mathbf{0}$ (primer apartado del problema) son perpendiculares al vector director, y son independientes, por lo que *forman la base del subespacio pedido en el enunicado.*

... y por último la manera más corta que se me ocurre... por ¡eliminación Gauss-Jordan!...

$$[\boldsymbol{x}_a|\mathbf{I}] = \left[\begin{array}{c|ccccc}-2 & 1 & 0 & 0 & 0 & 0\\ 1 & 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 0 & 1\end{array}\right] \to \left[\begin{array}{c|ccccc}1 & -1/2 & 0 & 0 & 0 & 0\\ 1 & 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 0 & 1\end{array}\right] \to \left[\begin{array}{c|ccccc}1 & -1/2 & 0 & 0 & 0 & 0\\ 0 & 1/2 & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 0 & 1\end{array}\right] = [(\mathbf{I}_1)_{||}\mathbf{E}]$$

Las cuatro últimas filas de la matriz $\mathbf{E}$ (allí donde hay filas de ceros en $(\mathbf{E}_1)_{|-}$–que es la forma escalonada reducida de $\boldsymbol{x}_a$) son vectores perpendiculares a $\boldsymbol{x}_a$; y es evidente que son cuatro, y que son linealmente independientes, así que son una base del subespacio perpendicular a $\boldsymbol{x}_a$.

□

(Final September 10/11) Short questions set 1(a)

Es verdadero. Si $\mathbf{A}$ es simétrica, estonces $\mathbf{A}^\intercal = \mathbf{A}$, por tanto

$$\left(\mathbf{A}^2\right)^\intercal = \left(\mathbf{A}\mathbf{A}\right)^\intercal = \mathbf{A}^\intercal\mathbf{A}^\intercal = \left(\mathbf{A}^\intercal\right)^2 = \mathbf{A}^2,$$

es decir, que $\mathbf{A}^2$ también es simétrica.

□

(Final September 10/11) Short questions set 1(b)

Es verdadero. Veamoslo:

$$(\mathbf{I} - \mathbf{A})^2 = (\mathbf{I} - \mathbf{A})(\mathbf{I} - \mathbf{A}) = \mathbf{I} - \mathbf{A} - \mathbf{A} + \mathbf{A}^2 = \mathbf{I} - \mathbf{A} - \mathbf{A} + \mathbf{A} = \mathbf{I} - \mathbf{A}$$

A las matrices con esta propiedad se las denomina *"matrices idempotentes"*.

□

(Final September 10/11) Short questions set 1(c)

Es falso. El determinate de una matriz es igual al producto de sus autovaloes; si uno de ellos es cero, necesariamente la matriz es singular. En tal caso sus columnas son linealmente dependientes y es posible encontrar una solución distinta a la trivial ($\boldsymbol{x} = \mathbf{0}$) para dicho sistema homogeneo; así que hay más de una solución y el sistema es necesariamente *indeterminado*.

□

(Final September 10/11) Short questions set 1(d)

Verdadero. Por el mismo motivo de antes, $\underset{m\times m}{\mathbf{A}}$ es singular, lo que quiere decir que el subespacio generado por las columnas de $\mathbf{A}$ (*que llamaremos espacio columna de* $\mathbf{A}$, $\mathcal{C}(\mathbf{A})$) es de dimensión menor que m, pero eso quiere decir que existen vectores de $\mathbb{R}^m$ que no pertenencen a $\mathcal{C}(\mathbf{A})$. Si $\boldsymbol{b}$ fuera uno de ellos, entonces no existiría una combinación lineal de las columnas de $\mathbf{A}$ igual a $\boldsymbol{b}$, es decir, que $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$ será incompatible para dicho $\boldsymbol{b} \notin \mathcal{C}(\mathbf{A})$.

□

(Final September 10/11) Short questions set 1(e)

True. If $\mathbf{Q}$ is orthogonal, then $\mathbf{Q}^\intercal\mathbf{Q} = \mathbf{I}$; therefore the inverse of $\mathbf{Q}$ is its transpose ($\mathbf{Q}^\intercal = \mathbf{Q}^{-1}$), and then $\mathbf{Q}\mathbf{Q}^{-1} = \mathbf{I} = \mathbf{Q}\mathbf{Q}^\intercal$; but this means that the columns of $\mathbf{Q}^{-1}$ are orthogonal (since all the elements of $\mathbf{Q}\mathbf{Q}^\intercal = \mathbf{I}$ outside the main diagonal are zero) with norm equal to one (since $\mathbf{Q}\ \mathbf{Q}^\intercal$ has only ones in the main diagonal).

□

(Final September 10/11) Short questions set 1(f)

False. For the matrix

$$\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

1 is the only eigenvalue, but $\mathbf{A}$ is not the identity matrix.

□

(Final September 10/11) Short questions set 2(a)

2, puesto que las dos primeras son dependientes.

□

(Final September 10/11) Short questions set 2(b)

2 (= rango de $\mathbf{A}$)

□

(Final September 10/11) Short questions set 3(a)

La matriz simétrica asociada a dicha forma cuadrática es

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & a & 0 \\ 0 & 0 & 8 \end{bmatrix}$$

y sus menores principales son

$$\left|1\right| = 1; \qquad \begin{vmatrix} 1 & 1 \\ 1 & a \end{vmatrix} = a - 1; \qquad \begin{vmatrix} 1 & 1 & 0 \\ 1 & a & 0 \\ 0 & 0 & 8 \end{vmatrix} = 8(a-1);$$

Si $a = 1$ la matriz $\mathbf{Q}$ es semidefinida positiva (los signos son: +,0,0).

□

(Final September 10/11) Short questions set 3(b)

La matriz $\mathbf{Q}$ nunca puede ser definida negativa. Si $a < 1$ es no definida (signos: +,–,–). Si $a > 1$ es definida positiva (+,+,+).

□

(Final June 10/11) Exercise 1(a)

$$|\mathbf{A} - \lambda\mathbf{I}| = \begin{vmatrix} 2-\lambda & 1 & 0 & 0 \\ 1 & 2-\lambda & 0 & 0 \\ 0 & 0 & a-\lambda & 0 \\ 0 & 0 & a & a-\lambda \end{vmatrix} = (a-\lambda)^2 \begin{vmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} = 0.$$

Therefore, the eigenvalue $\lambda = a$ is repeated twice. We can get the other two eigenvalues solving

$$\begin{vmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} = (2-\lambda)^2 - 1 = 0; \qquad \Rightarrow \lambda^2 - 4\lambda + 3 = 0$$

Thus, the other two eigenvalues are 1 and 3.

□

(Final June 10/11) Exercise 1(b)

When $\lambda = a = 2$, the rank of the matrix

$$\mathbf{A} - 2\mathbf{I} = \begin{bmatrix} 2-\lambda & 1 & 0 & 0 \\ 1 & 2-\lambda & 0 & 0 \\ 0 & 0 & 2-\lambda & 0 \\ 0 & 0 & 2 & 2-\lambda \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \end{bmatrix}$$

is 3. Therefore $\dim \mathcal{N}\left(\mathbf{A}\right) = 1$ (only one free column); hence it is not possible to find two linearly independent eigenvectors for the repeated eigenvalue $\lambda = 2$. It follows that THE MATRIX IS NOT DIAGONALISABLE.

□

(Final June 10/11) Exercise 1(c)

$$|\mathbf{B} - \lambda\mathbf{I}| = \begin{vmatrix} 2-\lambda & 1 & 0 \\ 1 & 2-\lambda & 0 \\ 0 & 0 & 1-\lambda \end{vmatrix} = (1-\lambda)\cdot\left((2-\lambda)^2 - 1\right) = 0$$

Clearly one eigenvector is $\lambda = 1$. The other two are the roots of

$$\left((2-\lambda)^2 - 1\right) = 4 + \lambda^2 - 4\lambda - 1 = \lambda^2 - 4\lambda + 3 = 0.$$

that is, $\lambda = 3$ and $\lambda = 1$. Thus, $\boxed{\mathbf{D} = \left[\begin{smallmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right].}$

- For $\lambda = 3$
$$\mathbf{A} - 3\lambda\mathbf{I} = \begin{bmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & -2 \end{bmatrix};$$
therefore $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ is an eigenvector. Since its norm is $\sqrt{2}$, then $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ is a normalised eigenvector for $\lambda = 3$.
- For $\lambda = 1$
$$\mathbf{A} - \lambda\mathbf{I} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix};$$
because the last column of $(\mathbf{A} - \lambda\mathbf{I})$, the vector $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ is an eigenvector (with norm 1); besides, from the other two columns, it is ease tho check that $\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$ is another eigenvector for $\lambda = 3$. Normalising the vector we get $\frac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$.

  It is not difficult to see that those three vector are orthogonal. Therefore:
$$\boxed{\mathbf{P} = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}.}$$

□

(Final June 10/11) Exercise 1(d)

The quadratic form is

$$f(x,y,z) = \boldsymbol{x}\mathbf{B}\boldsymbol{x} = \begin{pmatrix} x & y & z \end{pmatrix} \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 2x^2 + 2y^2 + z^2 + 2xy,$$

and we already know it is positive defined, since the three eigenvalues of the symmetric matrix $\mathbf{B}$ are positive.

$\square$

(Final June 10/11) Exercise 2(a)

$$\begin{bmatrix}1&4&2&3\\2&3&3&7\\0&1&0&3\\0&2&0&a\end{bmatrix}\to\begin{bmatrix}1&4&2&3\\0&-5&-1&1\\0&1&0&3\\0&2&0&a\end{bmatrix}\to\begin{bmatrix}1&4&2&3\\0&1&0&3\\0&-5&-1&1\\0&2&0&a\end{bmatrix}\to\begin{bmatrix}1&4&2&3\\0&1&0&3\\0&0&-1&16\\0&0&0&a-6\end{bmatrix}$$

In order to have a full rank matrix, the parameter a must be different from 6.

$\square$

(Final June 10/11) Exercise 2(b)

On the one hand

$$\det\mathbf{A}=\begin{vmatrix}1&4&2&3\\2&3&3&7\\0&1&0&3\\0&2&0&5\end{vmatrix}=1\begin{vmatrix}3&3&7\\1&0&3\\2&0&5\end{vmatrix}-2\begin{vmatrix}4&2&3\\1&0&3\\2&0&5\end{vmatrix}=-3\begin{vmatrix}1&3\\2&5\end{vmatrix}-2\cdot(-2)\begin{vmatrix}1&3\\2&5\end{vmatrix}=\begin{vmatrix}1&3\\2&5\end{vmatrix}=-1.$$

On the other hand

$$\begin{vmatrix}1&4&2&1\\2&3&3&0\\0&1&0&0\\0&2&0&0\end{vmatrix}=-1\cdot\begin{vmatrix}2&3&3\\0&1&0\\0&2&0\end{vmatrix}=-2\begin{vmatrix}1&0\\2&0\end{vmatrix}=0$$

Consequently, $x_4=\frac{0}{-1}=0$.

$\square$

(Final June 10/11) Exercise 2(c)

$$\left[\begin{array}{cccc|cccc}0&1&1&0&1&0&0&0\\1&0&0&1&0&1&0&0\\0&0&0&1&0&0&1&0\\0&1&0&0&0&0&0&1\end{array}\right]\to\left[\begin{array}{cccc|cccc}1&0&0&1&0&1&0&0\\0&1&1&0&1&0&0&0\\0&0&0&1&0&0&1&0\\0&1&0&0&0&0&0&1\end{array}\right]\to\left[\begin{array}{cccc|cccc}1&0&0&1&0&1&0&0\\0&1&1&0&1&0&0&0\\0&1&0&0&0&0&0&1\\0&0&0&1&0&0&1&0\end{array}\right]$$

$$\left[\begin{array}{cccc|cccc}1&0&0&1&0&1&0&0\\0&1&1&0&1&0&0&0\\0&0&-1&0&-1&0&0&1\\0&0&0&1&0&0&1&0\end{array}\right]\to\left[\begin{array}{cccc|cccc}1&0&0&1&0&1&0&0\\0&1&1&0&1&0&0&0\\0&0&1&0&1&0&0&-1\\0&0&0&1&0&0&1&0\end{array}\right]\to\left[\begin{array}{cccc|cccc}1&0&0&0&0&1&-1&0\\0&1&0&0&0&0&0&1\\0&0&1&0&1&0&0&-1\\0&0&0&1&0&0&1&0\end{array}\right]$$

Hence

$$\mathbf{B}^{-1}=\begin{bmatrix}0&1&-1&0\\0&0&0&1\\1&0&0&-1\\0&0&1&0\end{bmatrix}$$

and thus, multiplying by $\mathbf{B}^{-1}$ we get $\mathbf{B}\boldsymbol{x}=\boldsymbol{b}\quad\Rightarrow\quad\mathbf{B}^{-1}\mathbf{B}\boldsymbol{x}=\mathbf{B}^{-1}\boldsymbol{b}\quad\Rightarrow\quad\boldsymbol{x}=\mathbf{B}^{-1}\boldsymbol{b}$:

$$\boldsymbol{x}=\mathbf{B}^{-1}\boldsymbol{b}=\begin{bmatrix}0&1&-1&0\\0&0&0&1\\1&0&0&-1\\0&0&1&0\end{bmatrix}\begin{pmatrix}1\\0\\0\\0\end{pmatrix}=\begin{pmatrix}0\\0\\1\\0\end{pmatrix}.$$

$\square$

(Final June 10/11) Exercise 3(a)

$$\begin{bmatrix}2&1&2\\4&1&2\\2&1&a\end{bmatrix}\to\begin{bmatrix}2&1&2\\0&-1&-2\\0&0&a-2\end{bmatrix}$$

There is only one solution when the matrix is full rank, that is, when $a\neq 2$.

$\square$

(Final June 10/11) Exercise 3(b)

Since the coeficiente matrix $\underset{3\times 3}{\mathbf{A}}$ has rank 2, there is only one free variable. Therefore, $\dim \mathcal{N}\left(\mathbf{A}\right) = 1$. Because the third column is twice the second one, we can choose either y or z as free variables.

Performing the Gauss-Jordan elimination, we get the reduced echelon form:

$$\begin{bmatrix} 2 & 1 & 2 \\ 0 & -1 & -2 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 1/2 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}.$$

Thus the vector $\begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix}$ is a basis of $\mathcal{N}\left(\mathbf{A}\right)$.

The full set of solutions consist of all multiples of the vector in the basis:

$$\text{a solution vector is any } \boldsymbol{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \text{ multiple of } \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix};$$

that is $\boldsymbol{x} = a \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix}$ for any $a \in \mathbb{R}$.

□

(Final June 10/11) Exercise 3(c)

The point(1,1,1) is solution to the non-linear system.

$$\begin{cases} 1^2 + \frac{1^2}{2} + 4\sqrt{1} & = 5.5 \\ 2 \cdot 1^2 + 1 + 2 \cdot 1 & = 5 \end{cases}.$$

The Jacobian matrix is

$$\begin{bmatrix} 2x & y & 2z^{-1/2} \\ 4x & 1 & 2 \end{bmatrix} \xrightarrow{\text{evaluating at (1,1,1)}} \begin{bmatrix} 2 & 1 & 2 \\ 4 & 1 & 2 \end{bmatrix} \xrightarrow{\text{by Gaussian elimination}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \end{bmatrix}$$

Hence

$$\begin{pmatrix} x \\ y \end{pmatrix} \approx \begin{pmatrix} 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 0 \\ 2 \end{pmatrix} \Delta z = \begin{pmatrix} 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 0 \\ 2 \end{pmatrix} 0.1 = \begin{pmatrix} 1 \\ 0.8 \end{pmatrix}$$

Therefore, when we evaluate the non linear equations at $\begin{pmatrix} 1 \\ 0.8 \\ 1.1 \end{pmatrix}$ the system is close to $\begin{pmatrix} 5.5 \\ 5 \end{pmatrix}$.

□

(Final June 10/11) Exercise 4(a)

Since the right hand side vector $\boldsymbol{b}$ belongs to $\mathbb{R}^3$, then $\mathbf{A}$ has three rows. In addition, $\boldsymbol{x}$ also belongs to $\mathbb{R}^3$, thus $\mathbf{A}$ has also three columns.

Besides, there are two special solutions; therefore $\operatorname{rg}\left(\mathbf{A}\right) = 3 - 2 = 1$. It follows that there is only one pivot row, hence $\dim \mathcal{C}\left(\mathbf{A}^\mathsf{T}\right) = 1$.

□

(Final June 10/11) Exercise 4(b)

From the particular solution, it follows that twice the first column equals the right hand side vector $\begin{pmatrix} 2 \\ 4 \\ 2 \end{pmatrix}$, hence, the first column of $\mathbf{A}$ is $\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$.

Because the rank is one, the other columns are multiples of the first one. From the first special solution we know that the second column must be the opposite of the first one, or $\begin{pmatrix} -1 \\ -2 \\ -1 \end{pmatrix}$. Finally, from the second special solution it follows that the last column is the zero vector. Consequently,

$$\mathbf{A} = \begin{bmatrix} 1 & -1 & 0 \\ 2 & -2 & 0 \\ 1 & -1 & 0 \end{bmatrix}.$$

$\square$

(Final June 10/11) Exercise 4(c)

For any vector $\boldsymbol{b}$ in the column space of $\mathbf{A}$; in other words, the system is solvable for any vector $\boldsymbol{b} = a \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$; where a is any real number (for any multiple of the first column).

$\square$

(Final June 10/11) Short questions set 1(a)

$|\mathbf{AB}^2| = 2 \cdot (-2)^2 = 8.$
$|(\mathbf{AB})^{-1}| = \frac{1}{|\mathbf{AB}|} = \frac{1}{-4}.$

$\square$

(Final June 10/11) Short questions set 1(b)

There is no enough information to compute the determinant of $\mathbf{A} + \mathbf{B}$; but, since $|\mathbf{AB}|$ =-4, we know $\mathbf{AB}$ is a full rank matrix; therefore its rank is 3.
3×3

$\square$

(Final June 10/11) Short questions set 2(a)

There are two cases:

- $a = -4/5$ and $b = 3/5$
- $a = 4/5$ and $b = -3/5$.

$\square$

(Final June 10/11) Short questions set 2(b)

Any values of a and b such as the first column is not a multiple of the second; for example, $a = 1$ and $b = 0$.

$\square$

(Final June 10/11) Short questions set 2(c)

This is just the opposite case..., here we need a singular matrix; therefore we can use any multiple of the second column; for example: $a = 3$ and $b = 4$.

$\square$

(Final June 10/11) Short questions set 2(d)

By symmetry, $b = 3/5$. In addition, we need $a < 0$ and $\det \mathbf{A} > 0$; that is $a \cdot 4/5 - (3/5)^2 > 0$, or

$$a \cdot 4/5 > (3/5)^2$$

something impossible if $a < 0$. Therefore, THERE ISN'T SUCH VALUES OF a AND b.

$\square$

(Final June 10/11) Short questions set 3(a)

We need a rank 3 matrix; by Gaussian elimination we get:

$$\begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 1 & 2 & 3 \\ a & 1 & 1 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 2 \\ a & 1 & 1 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 1 & 1-a & 2-a \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & -a & -a \end{bmatrix},$$

therefore, if $a \neq 0$ the rank of $\mathbf{A}$ is 3, and the dimension of $\mathcal{N}\left(\mathbf{A}\right)$ is one.

$\square$

(Final June 10/11) Short questions set 3(b)

When $a = 0$; in that case $\dim \mathcal{N}\left(\mathbf{A}\right) = 2$.

$\square$

(Final June 10/11) Short questions set 4(a)

Since the first two vectors are the same, the dimension of $\mathcal{N}\left(\mathbf{A}\right)$ is 2. The number of the columns is 4, therefore the rank of $\mathbf{A}$ is 2.

The last vector is telling us that the last column of $\mathbf{A}$ is zero vector; and the first vector means that the first column of $\mathbf{A}$ is the opposite of the third. Then, one possibility is:

$$\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad \Longrightarrow \qquad \begin{cases} x - z & = 0 \\ y & = 0 \end{cases}.$$

But that is not the only possible answer; for example we can add zero rows below.

The coefficient matrix $\underset{m\times 4}{\mathbf{A}}$ must be rank 2, with a fourth column full of zeros, and a first column opposite to the third one.

□

(Final June 10/11) Short questions set 4(b)

Since $\mathbf{A}$ has a characteristic polynomial of degree 5, we know that $\mathbf{A}$ is a 5×5 matrix. Since 0 is not a root of $p(\cdot)$ and so is not an eigenvalue, we know $\mathbf{A}$ is invertible so $rank(A) = 5$.

□

(Final September 09/10) Exercise 1(a)

Los autovalores de

$$\begin{bmatrix}3&1&1\\0&3&1\\0&0&2\end{bmatrix}$$

son los elementos de la diagonal; $\lambda = 3$ (con multiplicidad 2) y $\lambda = 2$. Pero el rango de

$$\mathbf{A} - 3\lambda = \begin{bmatrix}3-3&1&1\\0&3-3&1\\0&0&2-3\end{bmatrix}$$

es 2; por tanto la matriz no es diagonalizable.

□

(Final September 09/10) Exercise 1(b)

Los autovalores de

$$\begin{bmatrix}2&1&1\\0&3&1\\0&0&2\end{bmatrix}$$

son los elementos de la diagonal; $\lambda = 3$ y $\lambda = 2$ (con multiplicidad 2). El rango de

$$\mathbf{A} - 2\lambda = \begin{bmatrix}2-2&1&1\\0&3-2&1\\0&0&2-2\end{bmatrix} = \begin{bmatrix}0&1&1\\0&1&1\\0&0&0\end{bmatrix}$$

es 1; por tanto la matriz es diagonalizable.

Dos autovectores independientes correspondientes al autovalor $\lambda = 2$ son $\begin{pmatrix}1\\0\\0\end{pmatrix}$ y $\begin{pmatrix}0\\1\\-1\end{pmatrix}$.

Por otra parte

$$\mathbf{A} - 3\lambda = \begin{bmatrix}2-3&1&1\\0&3-3&1\\0&0&2-3\end{bmatrix} = \begin{bmatrix}-1&1&1\\0&0&1\\0&0&-1\end{bmatrix}$$

Un autovector correspondiente al autovalor $\lambda = 3$ es $\begin{pmatrix}1\\1\\0\end{pmatrix}$.

Así pues

$$\mathbf{D} = \begin{bmatrix}2&&\\&2&\\&&3\end{bmatrix} \quad \text{and} \quad \mathbf{S} = \begin{bmatrix}1&0&1\\0&1&1\\0&-1&0\end{bmatrix}$$

son matrices tales que $\mathbf{A} = \mathbf{SDS}^{-1}$.

□

(Final September 09/10) Exercise 1(c)

Sea como sea $\mathbf{A}$, la matriz $\mathbf{A}^\intercal\mathbf{A}$ siempre es simétrica; y por tanto es diagonalizable, y es posible encontrar una base ortonormal de autovectores de $\mathbf{A}^\intercal\mathbf{A}$.

□

(Final September 09/10) Exercise 1(d)

Basta con encontrar los valores de a que hacen la matriz de rango completo; es decir, cualquier valor de a distinto de cero ($a \neq 0$) (para que la matriz sea invertible) y simultáneamente distinto de tres ($a \neq 3$) (para que la matriz sea diagonalizable).

□

(Final September 09/10) Exercise 2(a)

$$\left[\begin{array}{cccc|c} 1 & 2 & -1 & 1 & -1 \\ -1 & -2 & 3 & 5 & -5 \\ -1 & -2 & -1 & -7 & 7 \end{array}\right] \xrightarrow{\mathsf{E}_{21}(1)} \left[\begin{array}{cccc|c} 1 & 2 & -1 & 1 & -1 \\ 0 & 0 & 2 & 6 & -6 \\ -1 & -2 & -1 & -7 & 7 \end{array}\right] \xrightarrow{\mathsf{E}_{31}(1)}$$
$$\left[\begin{array}{cccc|c} 1 & 2 & -1 & 1 & -1 \\ 0 & 0 & 2 & 6 & -6 \\ 0 & 0 & -2 & -6 & 6 \end{array}\right] \xrightarrow{\mathsf{E}_{23}(1)} \left[\begin{array}{cccc|c} 1 & 2 & -1 & 1 & -1 \\ 0 & 0 & 2 & 6 & -6 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Tanto la matriz de coeficientes, cómo la matriz ampliada tienen rango 2.

□

(Final September 09/10) Exercise 2(b)

$$\left[\begin{array}{cccc|c} 1 & 2 & -1 & 1 & -1 \\ 0 & 0 & 2 & 6 & -6 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{\mathsf{E}_{22}(1/2)} \left[\begin{array}{cccc|c} 1 & 2 & -1 & 1 & -1 \\ 0 & 0 & 1 & 3 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{\mathsf{E}_{12}(1)} \left[\begin{array}{cccc|c} 1 & 2 & 0 & 4 & -4 \\ 0 & 0 & 1 & 3 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Una solución particular es

$$\boldsymbol{x}_p = \begin{pmatrix} -4 \\ 0 \\ -3 \\ 0 \end{pmatrix}, \quad \text{es decir,} \quad \begin{matrix} x_1 = -4 \\ x_2 = 0 \\ x_3 = -3 \\ x_4 = 0 \end{matrix}$$

Solución al sistema homogeneo es cualquier combinación de los vectores

$$\boldsymbol{x}_a = \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} \quad \text{y} \quad \boldsymbol{x}_b = \begin{pmatrix} -4 \\ 0 \\ -3 \\ 1 \end{pmatrix}, \quad \text{es decir,} \quad \begin{matrix} x_1 = -2a - 4b \\ x_2 = a \\ x_3 = -3b \\ x_4 = b \end{matrix}$$

para cualesquiera valores a y b.

Así pues, la solución al sistema propuesto es de la forma

$$\boldsymbol{x} = \begin{pmatrix} -4 \\ 0 \\ -3 \\ 0 \end{pmatrix} + a \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + b \begin{pmatrix} -4 \\ 0 \\ -3 \\ 1 \end{pmatrix}; \quad \text{o bien,} \quad \begin{matrix} x_1 = -4 - 2a - 4b \\ x_2 = a \\ x_3 = -3 - 3b \\ x_4 = b \end{matrix}$$

para cualesquiera valores a y b.

□

(Final September 09/10) Exercise 2(c)

Es un **plano** paralelo al generado por las combinaciones lineales de $\boldsymbol{x}_a$ y $\boldsymbol{x}_b$ (que es la solución del sistema homogeneo) pero que pasa por el punto $\boldsymbol{x}_p = (-7, 0, -6, 0)^{\intercal}$ (que es uno de los infinitos vectores que resuelven el sistema completo).

□

(Final September 09/10) Exercise 3(a)

$$\begin{vmatrix} a-2 & 1 & 2 \\ b-4 & 3 & 4 \\ c-6 & 5 & 6 \end{vmatrix} = \begin{vmatrix} a & 1 & 2 \\ b & 3 & 4 \\ c & 5 & 6 \end{vmatrix} = 3$$

□

(Final September 09/10) Exercise 3(b)

$$\begin{vmatrix} 7a & 7 & 14 \\ b & 3 & 4 \\ c & 5 & 6 \end{vmatrix} = 7 \begin{vmatrix} a & 1 & 2 \\ b & 3 & 4 \\ c & 5 & 6 \end{vmatrix} = 7 \times 3 = 21$$

□

(Final September 09/10) Exercise 3(c)

$$\left|(2\mathbf{A})^{-1}\mathbf{A}^{\mathsf{T}}\right| = \frac{1}{\det 2\mathbf{A}} \det \mathbf{A} = \frac{1}{2^3 \det \mathbf{A}} \det \mathbf{A} = \frac{1}{8}.$$

□

(Final September 09/10) Exercise 3(d)

$$\begin{vmatrix} a-2 & 1 & 2 \\ b & 3 & 4 \\ c & 5 & 6 \end{vmatrix} = \begin{vmatrix} a & 1 & 2 \\ b & 3 & 4 \\ c & 5 & 6 \end{vmatrix} - \begin{vmatrix} 2 & 1 & 2 \\ 0 & 3 & 4 \\ 0 & 5 & 6 \end{vmatrix} = 3 + 4 = 7$$

□

(Final September 09/10) Short questions set 1(a)

No siempre tiene solución; por ejemplo:

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 2 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

Nótese que el rango de la matriz de coeficientes $\mathbf{A}$ es uno, pero el rango de la matriz ampliada $[\mathbf{A}|\boldsymbol{b}]$ es dos.

□

(Final September 09/10) Short questions set 1(b)

Puesto que el sistema tiene más variables que ecuaciones, cuando el sistema tiene solución, ésta núnca puede ser única.

□

(Final September 09/10) Short questions set 1(c)

Que $\boldsymbol{b}$ sea combinación lineal de las columnas de $\mathbf{A}$; es decir, que la matriz de coeficientes $\mathbf{A}$, y la matriz ampliada $[\mathbf{A}|\boldsymbol{b}]$ tengan el mismo rango.

□

(Final September 09/10) Short questions set 1(d)

Puesto que el $\boldsymbol{b}$ tiene tres componentes (pertenece a $\mathbb{R}^3$), la condición es que el rango de $\mathbf{A}$ sea tres.

□

(Final September 09/10) Short questions set 2(a)

$$\left[\begin{array}{cccc|cccc} 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -2 & 0 & 0 & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{cccc|cccc} 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1/2 \end{array}\right] \rightarrow \left[\begin{array}{cccc|cccc} 1 & 1 & 0 & 0 & 1 & 0 & 0 & 1/2 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & -1/2 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1/2 \end{array}\right]$$
$$\rightarrow \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & -1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & -1/2 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1/2 \end{array}\right]$$

Así pues,

$$\mathbf{A}^{-1} = \begin{bmatrix} 1 & -1 & 0 & 1 \\ 0 & 1 & 0 & -1/2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1/2 \end{bmatrix}$$

□

(Final September 09/10) Short questions set 2(b)

$||\boldsymbol{v}||^2 = \boldsymbol{v} \cdot \boldsymbol{v} = 4 + 1 + 0 + 16 + 4 = 25$ so we take $\boldsymbol{u} = \boldsymbol{v}/||\boldsymbol{v}|| = (2/5, -1/5, 0.4/5, -2/5)$.

□

(Final September 09/10) Short questions set 2(c)

$$q(x,y,z) = x^2 + 6xy + y^2 + az^2 = \begin{pmatrix} x & y & z \end{pmatrix} \begin{bmatrix} 1 & 3 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & a \end{bmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

Puesto que $|1| > 0$, y $\begin{vmatrix}1 & 3\\ 3 & 1\end{vmatrix} < 0$; la forma cuadrática no es definida (sea cual sea el valor de a).

□

(Final September 09/10) Short questions set 2(d)

$$\begin{vmatrix}0 & 0 & 0 & 3 & 0\\ -2 & 0 & 0 & 2 & 0\\ 8 & -1 & 0 & -7 & 2\\ -1 & 2 & 2 & 3 & 2\\ 2 & 2 & 3 & 6 & 4\end{vmatrix} = -3\begin{vmatrix}-2 & 0 & 0 & 0\\ 8 & -1 & 0 & 2\\ -1 & 2 & 2 & 2\\ 2 & 2 & 3 & 4\end{vmatrix} = (-3)\cdot(-2)\begin{vmatrix}-1 & 0 & 2\\ 2 & 2 & 2\\ 2 & 3 & 4\end{vmatrix} = (-3)\cdot(-2)\cdot(2) = 12$$

□

(Final September 09/10) Short questions set 2(e)

$$\left[\mathbf{A}^3\right]\boldsymbol{v} = \left[\begin{bmatrix}2 & 2\\ 0 & -1\end{bmatrix}\begin{bmatrix}2 & \\ & -2\end{bmatrix}^3\begin{bmatrix}2 & 2\\ 0 & -1\end{bmatrix}^{-1}\right]\begin{pmatrix}1\\ -1\end{pmatrix} = \left[\begin{bmatrix}2 & 2\\ 0 & -1\end{bmatrix}\begin{bmatrix}2^3 & \\ & -2^3\end{bmatrix}\begin{bmatrix}1/2 & 1\\ 0 & -1\end{bmatrix}\right]\begin{pmatrix}1\\ -1\end{pmatrix} =$$
$$= \begin{bmatrix}8 & 32\\ 0 & -8\end{bmatrix}\begin{pmatrix}1\\ -1\end{pmatrix} = \begin{pmatrix}-24\\ 8\end{pmatrix}.$$

□

(Final September 09/10) Short questions set 2(f)

If $\mathbf{A}^\intercal = 2\mathbf{A}$, then also $\mathbf{A} = 2\mathbf{A}^\intercal = 2(2\mathbf{A}) = 4\mathbf{A}$ so $\mathbf{A} = \mathbf{0}$; and, of course, the rows of $\mathbf{A}$ are then linearly dependent.

□

(Final June 09/10) Exercise 1(a)

The rank is 4 (there are 4 pivots in the reduced row echelon form of $\mathbf{A}$).

□

(Final June 09/10) Exercise 1(b)

$$\boldsymbol{x} = a\begin{pmatrix}1\\1\\0\\0\\0\\0\\0\end{pmatrix} + b\begin{pmatrix}-2\\0\\-1\\1\\0\\0\\0\end{pmatrix} + c\begin{pmatrix}1\\0\\-1\\0\\0\\1\\0\end{pmatrix} \qquad \text{for } a, b, c, d \text{ in } \mathbb{R}.$$

□

(Final June 09/10) Exercise 1(c)

$$\boldsymbol{x} = x_2\begin{pmatrix}1\\1\\0\\0\\0\\0\\0\end{pmatrix} + x_4\begin{pmatrix}-2\\0\\-1\\1\\0\\0\\0\end{pmatrix} + x_6\begin{pmatrix}1\\0\\-1\\0\\0\\1\\0\end{pmatrix} = \begin{pmatrix}x_2 - 2x_4 + x_6\\ x_2\\ -x_4 - x_6\\ x_4\\ 0\\ x_6\\ 0\end{pmatrix}$$

□

(Final June 09/10) Exercise 1(d)

No, since $\mathcal{C}\left(\mathbf{A}\right) = \mathbb{R}^4$ then $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$ has solution for any vector $\boldsymbol{b}$ in $\mathbb{R}^4$.

□

(Final June 09/10) Exercise 1(e)

$$\boldsymbol{v} = \begin{pmatrix}1\\2\\-3\\0\end{pmatrix}$$

□

(Final June 09/10) Exercise 2(a)

Since the matrix is triangular, the eigenvalues are the numbers on the main diagonal: $\lambda_1 = 1$ and $\lambda_2 = 2$.

For $\lambda = 1$

$$(\mathbf{A} - \lambda \mathbf{I}) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix}$$

The following are three linearly independent eigenvectors

$$\boldsymbol{x}_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}; \quad \boldsymbol{x}_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}; \quad \boldsymbol{x}_3 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

For $\lambda = 2$

$$(\mathbf{A} - 2\lambda \mathbf{I}) = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 1 & 1 & 0 & -1 \end{bmatrix}$$

The following is an eigenvector

$$\boldsymbol{x}_4 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}$$

□

(Final June 09/10) Exercise 2(b)

Yes, since there are 4 linearly independent eigenvectors

□

(Final June 09/10) Exercise 2(c)

This factorization $\mathbf{A} = \mathbf{PDP^T}$ implies that $\mathbf{A}$ must be symetric; but $\mathbf{A}$ is not. Therefore, it is not possible.

□

(Final June 09/10) Exercise 2(d)

$$|\mathbf{A}^{-1}| = \frac{1}{|\mathbf{A}|} = \frac{1}{\text{product of eigenvalues of } \mathbf{A}} = \frac{1}{2}.$$

□

(Final June 09/10) Exercise 3(a)

$$\left[\begin{array}{ccc|c} 1 & -1 & 2 & 1 \\ 2 & -3 & m & 3 \\ -1 & 2 & 3 & 2m \end{array}\right] \xrightarrow{\mathbf{E}_{21}(-2)} \left[\begin{array}{ccc|c} 1 & -1 & 2 & 1 \\ 0 & -1 & m-4 & 1 \\ -1 & 2 & 3 & 2m \end{array}\right] \xrightarrow{\mathbf{E}_{31}(1)} \left[\begin{array}{ccc|c} 1 & -1 & 2 & 1 \\ 0 & -1 & m-4 & 1 \\ 0 & 1 & 5 & 2m+1 \end{array}\right] \xrightarrow{\mathbf{E}_{32}(1)}$$

$$\to \left[\begin{array}{ccc|c} 1 & -1 & 2 & 1 \\ 0 & -1 & m-4 & 1 \\ 0 & 0 & m+1 & 2m+2 \end{array}\right]$$

Since $2m + 2 = 0$ when $m + 1 = 0$, the system is always solvable, for any m.

□

(Final June 09/10) Exercise 3(b)

$$\left[\begin{array}{ccc|c} 1 & -1 & 2 & 1 \\ 0 & -1 & -5 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 7 & 0 \\ 0 & 1 & 5 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

Un solución particular es $\boldsymbol{x}_p = \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}$, y las soluciones al sistema homogéneo son los múltiplos del vector $\boldsymbol{x}_n = \begin{pmatrix} -7 \\ -5 \\ 1 \end{pmatrix}$. Por tanto, la solución completa al sistema son todos los vectores que se pueden escribir como $\boldsymbol{x} = \boldsymbol{x}_p + a\boldsymbol{x}_n$ para cualquier número real a.

□

(Final June 09/10) Exercise 3(c)

El conjunto de puntos que son solución al sistema del apartado anterior es una recta en $\mathbb{R}^3$.

No es posible que el conjunto de soluciones sea un plano en ningún caso; para que fuera posible sería necesario que la matriz de coeficientes del sistema fuera de rango 1. Pero en este caso el rango es 2 para $m = -1$ o rango 3 cuando $m \neq -1$. En este último caso (rango 3), el conjunto de soluciones es un punto en $\mathbb{R}^3$.

□

(Final June 09/10) Exercise 3(d)

En este caso el sistema es

$$\left[\begin{array}{ccc|c} 1 & -1 & 2 & 1 \\ 0 & -1 & -3 & 1 \\ 0 & 0 & 2 & 4 \end{array}\right]$$

Procediendo a la sustitución hacia atrás tenemos

$$x_3 = 2; \quad \Rightarrow \quad x_2 = -7; \quad \Rightarrow \quad x_1 = 1 - 7 - 4 = -10;$$

Por tanto la solución en este caso es

$$\boldsymbol{x} = \begin{pmatrix} -10 \\ -7 \\ 2 \end{pmatrix}.$$

□

(Final June 09/10) Short questions set 1(a)

True, since

$$\mathbf{AB} = \mathbf{I} \quad \Rightarrow \quad \mathbf{B} = \mathbf{A}^{-1}.$$

and

$$\mathbf{CA} = \mathbf{I} \quad \Rightarrow \quad \mathbf{C} = \mathbf{A}^{-1}.$$

Therefore $\mathbf{B}$ and $\mathbf{C}$ are the same matrix $\mathbf{A}^{-1}$.

□

(Final June 09/10) Short questions set 1(b)

False:

$$(\mathbf{AB})^2 = (\mathbf{AB})(\mathbf{AB}) = \mathbf{ABAB}$$

is in general different from

$$\mathbf{A}^2\mathbf{B}^2 = \mathbf{AABB}.$$

Example:

$$\mathbf{A} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}; \quad \mathbf{B} = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}. \qquad \mathbf{ABAB} = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}; \quad \mathbf{AABB} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

□

(Final June 09/10) Short questions set 1(c)

True, since

$$|\mathbf{AA}^\intercal| = |\mathbf{A}||\mathbf{A}^\intercal| = |\mathbf{A}||\mathbf{A}| = |\mathbf{A}|^2.$$

□

(Final June 09/10) Short questions set 2(a)

Yes, it is. A 3 by 3 matrix with 3 different eigenvalues.

□

(Final June 09/10) Short questions set 2(b)

No, it is not. Since $\boldsymbol{v}_3 = -\boldsymbol{v}_1$, then $\boldsymbol{v}_3$ must be an eigenvector associated to λ_1.

$\square$

(Final June 09/10) Short questions set 2(c)

$$\mathbf{A}(\boldsymbol{v}_1 - \boldsymbol{v}_2)) = \mathbf{A}\boldsymbol{v}_1 - \mathbf{A}\boldsymbol{v}_2 = \lambda_1\boldsymbol{v}_1 - \lambda_2\boldsymbol{v}_2 = 1\begin{pmatrix}1\\0\\1\end{pmatrix} - 2\begin{pmatrix}1\\1\\1\end{pmatrix} = \begin{pmatrix}-1\\-2\\-1\end{pmatrix}$$

$\square$

(Final June 09/10) Short questions set 3(a)

Lets check if the rank is 4:

$$\mathbf{A} = \begin{bmatrix}0 & 1 & a & 1\\1 & a & 0 & 0\\0 & 0 & 1 & a\\2 & 2a & 0 & 1\end{bmatrix} \xrightarrow[{[1\rightleftharpoons 2]}]{\tau} \begin{bmatrix}1 & a & 0 & 0\\0 & 1 & a & 1\\0 & 0 & 1 & a\\2 & 2a & 0 & 1\end{bmatrix} \xrightarrow{\underset{[(-2)1+3]}{\tau}} \begin{bmatrix}1 & a & 0 & 0\\0 & 1 & a & 1\\0 & 0 & 1 & a\\0 & 0 & 0 & 1\end{bmatrix} = \mathbf{U}$$

This matrix has rank 4 for any value a; therefore it is invertible.

$\square$

(Final June 09/10) Short questions set 3(b)

$$\left[\begin{array}{cccc|cccc}0 & 1 & 0 & 1 & 1 & 0 & 0 & 0\\1 & 0 & 0 & 0 & 0 & 1 & 0 & 0\\0 & 0 & 1 & 0 & 0 & 0 & 1 & 0\\2 & 0 & 0 & 1 & 0 & 0 & 0 & 1\end{array}\right] \xrightarrow[{[1\rightleftharpoons 2]}]{\tau} \left[\begin{array}{cccc|cccc}1 & 0 & 0 & 0 & 0 & 1 & 0 & 0\\0 & 1 & 0 & 1 & 1 & 0 & 0 & 0\\0 & 0 & 1 & 0 & 0 & 0 & 1 & 0\\2 & 0 & 0 & 1 & 0 & 0 & 0 & 1\end{array}\right] \xrightarrow{\underset{[(-2)1+3]}{\tau}}$$

$$\rightarrow \left[\begin{array}{cccc|cccc}1 & 0 & 0 & 0 & 0 & 1 & 0 & 0\\0 & 1 & 0 & 1 & 1 & 0 & 0 & 0\\0 & 0 & 1 & 0 & 0 & 0 & 1 & 0\\0 & 0 & 0 & 1 & 0 & -2 & 0 & 1\end{array}\right] \xrightarrow[{[(-1)4+2]}]{\tau} \left[\begin{array}{cccc|cccc}1 & 0 & 0 & 0 & 0 & 1 & 0 & 0\\0 & 1 & 0 & 0 & 1 & 2 & 0 & -1\\0 & 0 & 1 & 0 & 0 & 0 & 1 & 0\\0 & 0 & 0 & 1 & 0 & -2 & 0 & 1\end{array}\right]$$

Then,

$$\mathbf{A}^{-1} = \begin{bmatrix}0 & 1 & 0 & 0\\1 & 2 & 0 & -1\\0 & 0 & 1 & 0\\0 & -2 & 0 & 1\end{bmatrix}$$

$\square$

(Final June 09/10) Short questions set 4(a)

The corresponding matrix is

$$\begin{bmatrix}1 & -2 & 0\\-2 & 4 & 0\\0 & 0 & 5\end{bmatrix}.$$

Expanding $|\mathbf{A} - \lambda\mathbf{I}|$ along the last column:

$$\begin{vmatrix}1-\lambda & -2 & 0\\-2 & 4-\lambda & 0\\0 & 0 & 5-\lambda\end{vmatrix} = (1-\lambda)(4-\lambda)(5-\lambda) - 4(5-\lambda) = 0$$

It is clear that $\lambda = 5$ is an eigenvalue. Divinding the characteristic equation by $(5-\lambda)$ we get:

$$0 = (1-\lambda)(4-\lambda) - 4 = 4 - \lambda - 4\lambda + \lambda^2 - 4 = \lambda^2 - 5\lambda = 0$$

Therefore the two remaining roots are $\lambda = 0$ y $\lambda = 5$.

Two positive eigenvalues and one equal to zero: positive semi-definite matrix.

$\square$

(Final June 09/10) Short questions set 4(b)

The corresponding matrix is

$$\mathbf{A} = \begin{bmatrix}-1 & 1 & -a\\1 & 4 & 0\\-a & 0 & 1\end{bmatrix}.$$

We can alternatively prove that $-\mathbf{A}$ is positive definite; for example checking the sines of its sub-determinants. And since $\begin{vmatrix}1\end{vmatrix} = 1$, but $\begin{vmatrix}1 & -1\\-1 & -4\end{vmatrix} = -5$. Then $(-\mathbf{A})$ is not definite, neither $\mathbf{A}$.

$\square$