Mathematics II

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You can find the last version of these course materials at

https://github.com/mbujosab/MatematicasII/tree/main/Eng

1 Highlights of **Lesson 6**

Highlights of Lesson 6

- Introduction to vector spaces and sub-spaces

2 Introduction

What are the main operations that we do with vectors?

- We add them: $\boldsymbol{v}+\boldsymbol{w}$
- We multiply them by numbers, usually called scalars: $\lambda\boldsymbol{v}$

3 Vector space: definition

A *vector space* is a set $\mathcal{V}$ together with two operations

Addition $(\vec{x} + \vec{y})$: $\mathcal{V} \times \mathcal{V} \to \mathcal{V}$

It associates with each pair $\vec{x}$, $\vec{y}$ another element of $\mathcal{V}$ called $\vec{x} + \vec{y}$

Scalar product $(\alpha \vec{x})$: $\mathbb{R} \times \mathcal{V} \to \mathcal{V}$

It associates with each pair α, $\vec{x}$ another element of $\mathcal{V}$ called $\alpha \vec{x}$

satisfying:

- $\vec{x} + \vec{y} = \vec{y} + \vec{x}$
- $\vec{x} + (\vec{y} + \vec{z}) = (\vec{x} + \vec{y}) + \vec{z}$
- There exists a unique $\vec{0}$ such that $\vec{x} + \vec{0} = \vec{x}$
- For each $\vec{x}$ there is a unique $-\vec{x}$ such that $\vec{x} + (-\vec{x}) = \vec{0}$
- $\alpha(\vec{x} + \vec{y}) = \alpha\vec{x} + \alpha\vec{y}$
- $(\alpha + \beta)\vec{x} = \alpha\vec{x} + \beta\vec{x}$
- $(\alpha \cdot \beta)\vec{x} = \alpha(\beta\vec{x})$
- $1\vec{x} = \vec{x}$

4 Vector space

- A vector space is a set of mathematical objects
  (they could be numbers, lists of numbers, matrices, functions, etc. . .)
- and two operations:
  - *vector addition*
  - *scalar multiplication*.

  satisfying the eigth above axioms.
- The elements of a vector space are called *vectors*.

For us, scalars will be always real numbers ($\mathbb{R}$).

5 Examples: $\mathbb{R}^2$

$\mathbb{R}^2$: The space of all vectors with two components

$$\begin{pmatrix} 3 \\ 2 \end{pmatrix}; \quad \begin{pmatrix} 0 \\ 0 \end{pmatrix}; \quad \begin{pmatrix} \pi \\ e \end{pmatrix}; \quad \begin{pmatrix} \text{1st comp.} \\ \text{2nd comp} \end{pmatrix}$$

$\mathbb{R}^2$ is represented by the usual xy plane

6 More examples

$\mathbb{R}^3$: the space af all vectors with *three* components

$$\begin{pmatrix} 3 \\ 2 \\ 0 \end{pmatrix}$$

$\mathbb{R}^1$: lists with only one real number: $(0,)$ $(\pi,)$ $(a,)$ $(7,)$

$\mathbb{R}^n$: the space af all vectors with n components

7 Subspaces

A subspace $\mathcal{W}$ of the vector space $\mathcal{V}$

is a non-empty subset of $\mathcal{V}$ (with the same operations of $\mathcal{V}$) such

that for any $\vec{v}$ and $\vec{w}$ in $\mathcal{W}$ and scalars c and d:

- $(\vec{v} + \vec{w})$ is in $\mathcal{W}$
- $(c \cdot \vec{v})$ is in $\mathcal{W}$

All linear combinations $(c \cdot \vec{v} + d \cdot \vec{w})$ stay in $\mathcal{W}$

$\mathcal{W} \subset \mathcal{V}$ is a subspace if it is closed under both operations.

A subspace of $\mathcal{V}$ is a vector space inside the vector space $\mathcal{V}$.

8 Examples

Which ones of the following subsets of $\mathbb{R}^2$ are subspaces?

- The first quarter-plane
- Any line in $\mathbb{R}^2$ through $(0, 0)$
- A line in $\mathbb{R}^2$ that doesn't contain the origin
- $\{\mathbf{0}\}$: the set that consists only of a zero vector $\mathbf{0}$

Every *subspace* has its own zero vector $\mathbf{0}$

9 List of all possible subspaces of $\mathbb{R}^2$

1. The whole space (plane) $\mathbb{R}^2$
2.
3.

and the subspaces of $\mathbb{R}^3$? gráfico 3D

10 Union and intersection of subspaces

Let $\mathcal{S}$ and $\mathcal{T}$ be two subspaces

- $\mathcal{S} \cup \mathcal{T}$: their union (take the vector in $\mathcal{S}$ and $\mathcal{T}$ all together)

  Is the union a subspace?

- $\mathcal{S} \cap \mathcal{T}$: the intersection (vectors belonging to both $\mathcal{S}$ and $\mathcal{T}$)

  Is the intersection a subspace? (proof?)

Questions of the Lecture 6

(L-6) QUESTION 1.

(a) Find a subset W in $\mathbb{R}^2$ ($W \subseteq \mathbb{R}^2$) closed under vector addition (if $\boldsymbol{v}, \boldsymbol{w} \in W$, then $\boldsymbol{v} + \boldsymbol{w} \in W$), but not under scalar multiplication ($c\boldsymbol{v}$ is not necessarily in W).
(b) Find a subset W in $\mathbb{R}^2$ ($W \subseteq \mathbb{R}^2$) closed under scalar multiplication (if $\boldsymbol{v}, \boldsymbol{w} \in W$, then $c\boldsymbol{v} \in W$) but not under vector addition ($\boldsymbol{v} + \boldsymbol{w} \in W$ is not necessarily in W).

(Strang, 2006, exercise 1 from section 2.1.)

(L-6) QUESTION 2. Consider $\mathbb{R}^2$ as a vector space. Which of the following are subspaces and which are not? If not, why not?

(a) $\{(a,\ a^2,) \mid a \in \mathbb{R}\}$
(b) $\{(b,\ 0,) \mid b \in \mathbb{R}\}$
(c) $\{(0,\ c,) \mid c \in \mathbb{R}\}$
(d) $\{(m,\ n,) \mid m, n \in \mathbb{Z}\}$ where $\mathbb{Z}$ is the set of integer numbers.
(e) $\{(d,\ e,) \mid d, e \in \mathbb{R},\ d \cdot e = 0\}$
(f) $\{(f,\ f,) \mid f \in \mathbb{R}\}$

(L-6) QUESTION 3. Why isn't $\mathbb{R}^2$ a subspace of $\mathbb{R}^3$?
(Strang, 2006, exercise 31 from section 2.1.)

(L-6) QUESTION 4. Let P be the plane in $\mathbb{R}^3$ defined by the equation

$$x - y - z = 3.$$

Find two vectors in P and show that their sum is not in P.

(L-6) QUESTION 5. Show that for any $\boldsymbol{b} \neq \boldsymbol{0}$, the solution set $\{\boldsymbol{x} \mid \mathbf{A}\boldsymbol{x} = \boldsymbol{b}\}$ does not form a subspace.

(L-6) QUESTION 6. Consider the set $\mathbb{R}^{2\times 2}$ as a vector space. Let

$$\mathbf{A} = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}; \qquad \mathbf{B} = \begin{bmatrix} 0 & 0 \\ 0 & -3 \end{bmatrix};$$

(a) Name a subspace containing $\mathbf{A}$ but not $\mathbf{B}$.
(b) Name a subspace containing $\mathbf{B}$ but not $\mathbf{A}$.
(c) Is there a subspace containing $\mathbf{A}$ and $\mathbf{B}$ but not the 2×2 identity matrix $\underset{n\times n}{\mathbf{I}}$?

(L-6) QUESTION 7. Consider the set $\mathbb{R}^{n\times n}$ as a vector space. Which of the following are subspaces?

(a) The symmetric matrices, $\mathcal{S} = \{\mathbf{A} \in \mathbb{R}^{n\times n} \mid {}_{j|}\mathbf{A} = \mathbf{A}_{|j}\}$
(b) The non-symmetric matrices, $\mathcal{NS} = \{\mathbf{A} \in \mathbb{R}^{n\times n} \mid \mathbf{A}^{\intercal} \neq \mathbf{A}\}$
(c) The skew-symmetric matrices, $\mathcal{AS} = \{\mathbf{A} \in \mathbb{R}^{n\times n} \mid \mathbf{A}^{\intercal} = -\mathbf{A}\}$

(L-6) QUESTION 8.

(a) The intersection of two planes through $(0,0,0,)$ is probably a____________, but it could be a ____________.
(b) The intersection of a plane through $(0,0,0,)$ with a line through $(0,0,0,)$ is probably a ____________, but it could be a ____________.
(c) If $\mathcal{S}$ and $\mathcal{T}$ are subspaces of $\mathbb{R}^5$, their intersection $\mathcal{S} \cap \mathcal{T}$ (vectors in both sub-spaces) is a subspace of $\mathbb{R}^5$. Check the requirements on $\boldsymbol{x} + \boldsymbol{y}$ and $c\boldsymbol{x}$.

(Strang, 2006, exercise 18 from section 2.1.)

(L-6) QUESTION 9. Which of the following subsets of $\mathbb{R}^3$ are actually subspaces?

(a) The plane of vectors $\boldsymbol{b} = (b_1,\ b_2,\ b_3,)$ with first component $b_1 = 0$.
(b) The plane of vectors $\boldsymbol{b} = (b_1,\ b_2,\ b_3,)$ with first component $b_1 = 1$.
(c) The vectors $\boldsymbol{b}$ with $b_2 b_3 = 0$ (this is the union of two subspaces. the plane $b_2 = 0$ and the plane $b_3 = 0$).
(d) The solitary vector $\boldsymbol{b} = \boldsymbol{0}$.
(e) All combinations of two given vectors $(1,\ 1,\ 0,)$ and $(2,\ 0,\ 1,)$.
(f) The vectors $(b_1,\ b_2,\ b_3,)$ that satisfies $b_3 - b_2 + 3b_1 = 0$.

(Strang, 2006, exercise 2 from section 2.1.)

One more. . . not so easy

(L-6) QUESTION 10. Addition and scalar multiplication are required to satisfy these eight rules:

1. $\boldsymbol{x} + \boldsymbol{y} = \boldsymbol{y} + \boldsymbol{x}$.
2. $\boldsymbol{x} + (\boldsymbol{y} + \boldsymbol{z}) = (\boldsymbol{x} + \boldsymbol{y}) + \boldsymbol{z}$.

3. There is a unique $\boldsymbol{0}$ ("zero vector") such that $\boldsymbol{x} + \boldsymbol{0} = \boldsymbol{x}$ for all $\boldsymbol{x}$.
4. For each $\boldsymbol{x}$ there is a unique vector $-\boldsymbol{x}$ such that $\boldsymbol{x} + (-\boldsymbol{x}) = \boldsymbol{0}$.
5. $1\boldsymbol{x} = \boldsymbol{x}$.
6. $(a \cdot b)\boldsymbol{x} = a(b\boldsymbol{x})$.
7. $a(\boldsymbol{x} + \boldsymbol{y}) = a\boldsymbol{x} + a\boldsymbol{y}$.
8. $(a + b)\boldsymbol{x} = a\boldsymbol{x} + b\boldsymbol{x}$.

(a) Suppose addition in $\mathbb{R}^2$ adds an extra 1 to each component, so that $(3,1,) + (5,0,) = (9,2,)$ instead of $(8,1,)$. With scalar multiplication unchanged, which rules are broken?
(b) Show that the set of all positive real numbers is a vector space, when the addition and multiplication are redefined to be as follows:

- $\boldsymbol{x} + \boldsymbol{y} = xy$
- $\boldsymbol{x} = x^c$

What is the "zero vector" $\boldsymbol{0}$?:
(c) Suppose $(x_1,\ x_2,) + (y_1,\ y_2,)$ is defined to be $\Big((x_1 + y_2),\ (x_2 + y_1),\Big)$; With the usual $c\boldsymbol{x} = (cx_1,\ cx_2,)$. which of the eight conditions are not satisfied?

(Strang, 1988, exercise 5 from section 2.1.)

1 Highlights of Lesson 7

Highlights of Lesson 7

- Null space of a matrix **A**: solving $\mathbf{A}\boldsymbol{x} = \mathbf{0}$

A natural algorithm for solving $\mathbf{A}\boldsymbol{x} = 0$?

by elimination (**column** reduction)

- **Column** (pre)echelon form
- pivot (*or endogenous*) variables and free (*or exogenous*) variables
- Special solutions

2 The subspaces of a matrix: the null space $\mathcal{N}(\mathbf{A})$

$$\mathbf{A}\boldsymbol{x} = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \\ 4 & 1 & 5 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$\mathcal{N}(\mathbf{A})$ is the set of vectors $\boldsymbol{x}$ that solve $\mathbf{A}\boldsymbol{x} = \mathbf{0}$.

$\mathcal{N}(\mathbf{A})$ is subset of ¿$\mathbb{R}^?$?

Find some solutions. Find all solutions

And what does it look like? (graph)

3 Is the null space $\mathcal{N}(\mathbf{A})$ a subspace?

We should check that the set of all solutions to $\mathbf{A}\boldsymbol{v} = \mathbf{0}$ is a subspace.
We should check that for any $a, b \in \mathbb{R}$

If $\mathbf{A}\boldsymbol{v} = \mathbf{0}$ and $\mathbf{A}\boldsymbol{w} = \mathbf{0}$, then

Therefore $\mathcal{N}(\mathbf{A})$ is a subspace

4 Solutions of a general linear system

Let me change the right-hand side to (one, two, three, four).

$$\mathbf{A}\boldsymbol{x} = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \\ 4 & 1 & 5 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$$

That is a very special right-hand side. And we know that there are some solutions

Do they form a subspace?
Is the zero vector $\mathbf{0}$ a solution?

5 A natural algorithm for finding $\mathcal{N}(\mathbf{A})$?

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 2 & 2 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 8 & 10 \end{bmatrix}$$

- Which columns are a linear combination of the other columns?
- Elimination will tell us. . .

6 Which columns are a linear combination of the other columns?

$$\begin{bmatrix} \mathbf{A} \\ \hline \mathbf{I} \end{bmatrix} = \left[\begin{array}{cccc} 1 & 2 & 2 & 2 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 8 & 10 \\ \hline 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{array}\right] \longrightarrow$$

Then $\mathbf{A}\begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \mathbf{0} \quad \Longrightarrow$

and $\mathbf{A}\begin{pmatrix} 2 \\ 0 \\ -2 \\ 1 \end{pmatrix} = \mathbf{0} \quad \Longrightarrow$

7 How to compute $\mathcal{N}(\mathbf{A})$: elimination and "special solutions"

$$\mathbf{A}\left(\mathbf{I}_{\tau_1 \cdots \tau_k}\right)_{|j} = \left(\mathbf{A}_{\tau_1 \cdots \tau_k}\right)_{|j}$$

$$\begin{bmatrix} \mathbf{A} \\ \hline \mathbf{I} \end{bmatrix} = \left[\begin{array}{cccc} 1 & 2 & 2 & 2 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 8 & 10 \\ \hline 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{array}\right] \xrightarrow[\substack{[(-2)\mathbf{1}+\mathbf{2}] \\ [(-2)\mathbf{1}+\mathbf{3}] \\ [(-2)\mathbf{1}+\mathbf{4}]}]{\tau} \left[\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 2 & 0 & 2 & 4 \\ 3 & 0 & 2 & 4 \\ \hline 1 & -2 & -2 & -2 \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{array}\right] \xrightarrow[{[(-2)\mathbf{3}+\mathbf{4}]}]{\tau} \left[\begin{array}{cccc} 1 & \mathbf{0} & 0 & \mathbf{0} \\ 2 & \mathbf{0} & 2 & \mathbf{0} \\ 3 & \mathbf{0} & 2 & \mathbf{0} \\ \hline 1 & -2 & -2 & 2 \\ & 1 & & 0 \\ & 0 & 1 & -2 \\ & 0 & & 1 \end{array}\right] = \begin{bmatrix} \mathbf{K} \\ \hline \mathbf{E} \end{bmatrix}$$

If $\mathbf{A}(\mathbf{E}_{|j}) = \mathbf{0}$ them $\mathbf{E}_{|j}$ is a solution to $\mathbf{A}\boldsymbol{x} = \mathbf{0}$

The number of pivots of **K** is the *rank* of a matrix

8 How to compute $\mathcal{N}(\mathbf{A})$: general solution

The general solution: $\mathcal{N}(\mathbf{A})$

What is the set of ALL solutions?

- $\mathcal{N}(\mathbf{A}) = \{ \quad \}$
- $\mathcal{N}(\mathbf{A}) = \left\{ \boldsymbol{x} \in \quad \right\}$

How many special solutions are there?

How many null columns are there?

9 Why aren't there more solutions?

Consider $\mathbf{A}x = \mathbf{0}$ and $\mathbf{AE} = \mathbf{K}$ $\quad (\mathbf{E} = \mathbf{I}_{\tau_1 \cdots \tau_k}$ fullrank$)$

Is x a combination of cols. of $\mathbf{E}$? $\quad (x = \mathbf{E}y)$

Using $y = \mathbf{E}^{-1}x$, we have that $x = \mathbf{E}y$
Do we need all columns of $\mathbf{E}$ to get x?

$$\left[\frac{\mathbf{A}}{\mathbf{I}}\right] \to \left[\frac{\mathbf{K}}{\mathbf{E}}\right] = \left[\begin{array}{ccccc} * & 0 & 0 & 0 & 0 \\ * & * & 0 & 0 & 0 \\ * & * & * & 0 & 0 \\ \hline \mathbf{E}_{|1} & \mathbf{E}_{|2} & \mathbf{E}_{|3} & \mathbf{E}_{|4} & \mathbf{E}_{|5} \end{array}\right]$$

$$\mathbf{A}x = \mathbf{AE}y = \mathbf{K}y = \mathbf{0} \Rightarrow (y_j = ? \text{ for pivot columns})$$

$\forall x \in \mathcal{N}(\mathbf{A})$, x is a combination of the *special solutions*

10 Computing $\mathcal{N}(\mathbf{A})$: complete algorithm for solving $\mathbf{A}x = \mathbf{0}$

An algorithm for solving $\mathbf{A}x = 0$

1. Find a pre-echelon form: $\left[\frac{\mathbf{A}}{\mathbf{I}}\right] \xrightarrow{\tau_1 \cdots \tau_k} \left[\frac{\mathbf{K}}{\mathbf{E}}\right]$
2. If there are *special solutions:*
   - Complete solution

     $$\mathcal{N}(\mathbf{A}) = \{\text{linear combinations of special solutions}\}$$
3. If no *special solutions*
   - Complete solution: $\mathcal{N}(\mathbf{A}) = \{\mathbf{0}\}$

11 Another example: $\mathcal{N}(\mathbf{A}^\intercal)$

$$\left[\frac{\mathbf{A}^\intercal}{\mathbf{I}}\right] = \left[\begin{array}{ccc} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 2 & 6 & 8 \\ 2 & 8 & 10 \\ \hline 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right] \xrightarrow[\substack{[(-2)\mathbf{1}+\mathbf{2}] \\ [(-3)\mathbf{1}+\mathbf{3}] \\ [(-1)\mathbf{2}+\mathbf{3}]}]{\tau} \qquad = \left[\frac{\mathbf{L}}{\mathbf{E}}\right]$$

How many pivots are there?

How many free columns $\qquad$ How many special solutions

set of solutions to $\mathbf{A}^\intercal x = \mathbf{0}$?

Questions of the Lecture 7

(L-7) Question 1. Reduce the matrices to a pre-echelon form, to find their ranks. Describe the nullspace with parametric equations.

(a) $\mathbf{A} = \begin{bmatrix} 1 & 2 & 0 & 5 \\ 2 & 3 & 1 & 4 \\ -1 & -1 & -1 & 1 \end{bmatrix}$.

(b) $\mathbf{F} = \begin{bmatrix} 1 & 2 & 1 \\ -1 & 3 & 4 \\ 2 & -1 & -3 \end{bmatrix}$

(c) $\mathbf{G} = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 3 & 1 \\ -2 & -1 & 4 \end{bmatrix}$.

(d) $\mathbf{H} = \begin{bmatrix} 1 & 3 \\ 2 & 1 \\ -1 & -3 \end{bmatrix}$.

(L-7) QUESTION 2. Describe the nullspace of the matrices with parametric equations

(a) $\mathbf{A} = \begin{bmatrix} 1 & 2 & -2 \\ 2 & 1 & 0 \end{bmatrix}$.

(b) $\mathbf{F} = \begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix}$.

(c) $\mathbf{G} = \begin{bmatrix} 1 & 2 & -4 \\ -1 & 1 & 3 \\ 1 & 5 & -5 \end{bmatrix}$.

(L-7) QUESTION 3. Reduce $\mathbf{A} = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{bmatrix}$ to pre-echelon form, to find their ranks. Find the special solutions to $\mathbf{A}\boldsymbol{x} = \mathbf{0}$. Find all solutions.
(Strang, 2006, exercise 2 from section 2.2.)

(L-7) QUESTION 4. Find a pre-echelon form and the rank of these matrices and the complete solution to the systems $\mathbf{A}\boldsymbol{x} = \mathbf{0}$:

(a) The 3 by 4 matrix of all ones.
(b) The 4 by 4 matrix with $a_{ij} = (-1)^{ij}$.
(c) The 3 by 4 matrix with $a_{ij} = (-1)^{j}$.

(Strang, 2006, exercise 13 from section 2.2.)

(L-7) QUESTION 5. The matrix **A** has two special solutions:

$$\boldsymbol{x}_1 = \begin{pmatrix} c \\ 1 \\ 0 \end{pmatrix}; \qquad \boldsymbol{x}_2 = \begin{pmatrix} d \\ 0 \\ 1 \end{pmatrix}$$

(a) Describe all the possibilities for the number of columns of **A**.
(b) Describe all the possibilities for the number of rows of **A**.
(c) Describe all the possibilities for the rank of **A**.

Briefly explain your answers.
(*MIT Course 18.06 Quiz 1, Fall, 2008*)

(L-7) QUESTION 6. Suppose **A** has column reduced echelon form **R**

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & \clubsuit \\ 2 & a & \clubsuit \\ 1 & 1 & \clubsuit \\ b & 8 & \clubsuit \end{bmatrix}, \quad \mathbf{R} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 2 & 0 \end{bmatrix}.$$

(a) What can you say about column 3 of **A**?
(b) What are the numbers a and b?
(c) Describe the nullspace of **A** if: $\mathbf{A} \begin{bmatrix} -1 & 2 & 1 \\ 1 & -1 & -2 \\ 0 & 0 & 1 \end{bmatrix} = \mathbf{R}$.

(L-7) QUESTION 7. Find the reduced column echelon form of these matrices

(a) $\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 1 & 2 & 3 \end{bmatrix}$.

(b) $\mathbf{B} = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 2 & 2 \\ 1 & 0 & 1 \end{bmatrix}$.

(c) $\mathbf{C} = \begin{bmatrix} 1 & 2 & 1 & 2 & 1 \\ 2 & 1 & 2 & 1 & 2 \\ 0 & 1 & 0 & 1 & 0 \end{bmatrix}$.

(d) $\mathbf{D} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \end{bmatrix}$.

(L-7) QUESTION 8. Consider the *invertible* matrix $\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

(a) Knowing **A** is invertible, and without any calculation ¿what is its reduced echelon form?
(b) Compute $\mathbf{A}^{-1}$.

(L-7) QUESTION 9. Consider the invertible matrix $\mathbf{A} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

(a) Without any computation, say what is its reduced echelon form.
(b) Find the inverse of **A**.

1 Highlights of Lesson 8

Highlights of Lesson 8

- The column space of a matrix **A**: solving $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$
- We will completely solve the *general* linear system $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$, ... if it has a solution
  - is there only one solution?
  - or is there a whole family of solutions?

$$\left\{ \boldsymbol{x} = \boldsymbol{x}_p + \boldsymbol{x}_n \;\middle|\; \begin{array}{l} \mathbf{A}(\boldsymbol{x}_p) = \boldsymbol{b} \\ \mathbf{A}(\boldsymbol{x}_n) = \mathbf{0} \end{array} \right\}$$

2 The subspaces of a matrix: the column space $\mathcal{C}(\mathbf{A})$

$$\mathbf{A} = \begin{bmatrix} 1 & 3 \\ 2 & 3 \\ 4 & 1 \end{bmatrix}$$

These column vectors are in

How do I fill the set out to be a subspace?... Taking...

This subspace is called *column space of* **A**: $\mathcal{C}(\mathbf{A})$.

So $\mathcal{C}(\mathbf{A})$ is a subspace of

3 The subspaces of a matrix: The column space $\mathcal{C}(\mathbf{A})$

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 2 & 2 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 8 & 10 \end{bmatrix}$$

$\mathcal{C}(\mathbf{A})$ is a subspace of

What is in $\mathcal{C}(\mathbf{A})$?

Is that space $\mathcal{C}(\mathbf{A})$ the whole three dimensional $\mathbb{R}^3$ space?

Let's connect this question with linear equations...

4 Link between $\mathcal{C}(\mathbf{A})$ and $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$

does $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$ have a solution for every $\boldsymbol{b}$? (the question for today)

$$\mathbf{A}\boldsymbol{x} = \begin{bmatrix} 1 & 2 & 2 & 2 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 8 & 10 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

Which right-hand sides allow me to solve this?

Can I find a solution for $\boldsymbol{b}_1 = (1, \ 2, \ 3,)$? and
$\boldsymbol{b}_2 = (2, \ 6, \ 8,)$? and $\boldsymbol{b}_3 = (0, \ 0, \ 0,)$? and
$\boldsymbol{b}_4 = (3, \ 6, \ 9,)$? and $\boldsymbol{b}_5 = (1, \ 0, \ 0,)$?

5 Link between $\mathcal{C}(\mathbf{A})$ and $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$

$$\mathbf{A}\boldsymbol{x} = \begin{bmatrix} 1 & 2 & 2 & 2 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 8 & 10 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

Can I throw away any column and keep the same? $\mathcal{C}(\mathbf{A})$?

and elimination will show which columns are linear combination of those to the left of it.

But how does Gaussian elimination affect $\mathcal{N}(\mathbf{A})$ and $\mathcal{C}(\mathbf{A})$?

6 In the next example we will use the reduced echelon form

Gauss-Jordan elimination: all pivots are 1, with zeros at the left

$$\left[\frac{\mathbf{A}}{\mathbf{I}}\right] = \left[\begin{array}{cccc} 1 & 2 & 2 & 2 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 8 & 10 \\ \hline 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow[{[2 \rightleftharpoons 3]}]{\substack{\boldsymbol{\tau} \\ [(-2)\mathbf{1}+\mathbf{2}] \\ [(-2)\mathbf{1}+\mathbf{3}] \\ [(-2)\mathbf{1}+\mathbf{4}] \\ [(-2)\mathbf{3}+\mathbf{4}]}} \left[\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 2 & 2 & 0 & 0 \\ 3 & 2 & 0 & 0 \\ \hline 1 & -2 & -2 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{\substack{\boldsymbol{\tau} \\ [(-1)\mathbf{2}+\mathbf{1}] \\ [(\frac{1}{2})\mathbf{2}]}} \left[\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ \hline 3 & -1 & -2 & 2 \\ 0 & 0 & 1 & 0 \\ -1 & \frac{1}{2} & 0 & -2 \\ 0 & 0 & 0 & 1 \end{array}\right] = \left[\frac{\mathbf{R}}{\mathbf{E}}\right]$$

$$\boxed{\begin{array}{l} \left(\mathbf{A}_{\tau_1\cdots\tau_k}\right)_{|j} = \mathbf{A}\left(\mathbf{I}_{\tau_1\cdots\tau_k}\right)_{|j} \\[2ex] \left(\mathbf{A}_{\tau_1\cdots\tau_k}\right)\left(\mathbf{I}_{\tau_k^{-1}\cdots\tau_1^{-1}}\right)_{|j} = \mathbf{A}_{|j} \end{array}} \implies \mathcal{C}(\mathbf{A}) = \mathcal{C}\left(\mathbf{A}_{\tau_1\cdots\tau_k}\right).$$

But in general $\mathcal{N}(\mathbf{A}) \neq \mathcal{N}\left(\mathbf{A}_{\tau_1\cdots\tau_k}\right)$.

7 Linear system of equations example

$$\begin{cases} x_1 + 2x_2 + 2x_3 + 2x_4 = b_1 \\ 2x_1 + 4x_2 + 6x_3 + 8x_4 = b_2 \\ 3x_1 + 6x_2 + 8x_3 + 10x_4 = b_3 \end{cases}$$

What is going to discover elimination about the columns?

What must $(b_1, b_2, b_3,)$ fulfil for a solution to exist?

If $b_1 = 1$ and $b_2 = 5$, what is the only b_3 that would be OK?
Let's see!

$$\mathbf{A}\boldsymbol{x} = \boldsymbol{b} \quad \Leftrightarrow \quad \mathbf{A}\boldsymbol{x} - 1\boldsymbol{b} = \mathbf{0} \quad \Leftrightarrow \quad \left[\ \mathbf{A} \ \middle|\ -\boldsymbol{b}\ \right] \begin{pmatrix} \boldsymbol{x} \\ 1 \end{pmatrix} = \mathbf{0}$$

8 Linear system of equations: condition for solvability

$$\left[\ \mathbf{A} \ \middle|\ -\boldsymbol{b}\ \right](\boldsymbol{x}, \quad 1,) = \mathbf{0}$$

$$\left[\begin{array}{c|c} \mathbf{A} & \textbf{-}\boldsymbol{b} \\ \hline \mathbf{I} & \mathbf{0} \\ \hline & 1 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & -b_1 \\ 0 & 1 & 0 & 0 & -b_2 \\ 1 & 1 & 0 & 0 & -b_3 \\ \hline 3 & -1 & -2 & 2 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ -1 & \frac{1}{2} & 0 & -2 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{\substack{\boldsymbol{\tau} \\ [(b_1)\mathbf{1}+\mathbf{5}] \\ [(b_2)\mathbf{2}+\mathbf{5}]}} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & b_1+b_2-b_3 \\ \hline 3 & -1 & -2 & 2 & 3b_1-b_2 \\ 0 & 0 & 1 & 0 & 0 \\ -1 & \frac{1}{2} & 0 & -2 & -b_1+\frac{1}{2}b_2 \\ 0 & 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 \end{array}\right] = \left[\begin{array}{c|c} \mathbf{R} & \mathbf{0} \\ \hline \mathbf{E} & \boldsymbol{x}_p \\ \hline & 1 \end{array}\right]$$

Then the condition for solvability is:

If $b_1 = 1$ and $b_2 = 5$ then $b_3 =$
If $\boldsymbol{b} = (1, \quad 5, \quad 6,)$ what is the last column when the system is solvable?
Solve for $\boldsymbol{b} = (2, \quad 2, \quad 4,)$

9 An algorithm to solve the linear system $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 2 & 2 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 8 & 10 \end{bmatrix}; \quad \boldsymbol{b} = \begin{pmatrix} 2 \\ 2 \\ 4 \end{pmatrix}; \qquad \mathbf{A}\boldsymbol{x} = \boldsymbol{b}$$

$$\left[\begin{array}{cccc|c} 1 & 2 & 2 & 2 & -2 \\ 2 & 4 & 6 & 8 & -2 \\ 3 & 6 & 8 & 10 & -4 \\ \hline 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 2 & 0 & 2 & 4 & 2 \\ 3 & 0 & 2 & 4 & 2 \\ \hline 1 & -2 & -2 & -2 & 2 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow[{[(-1)\mathbf{3}+\mathbf{5}]}]{\substack{\boldsymbol{\tau} \\ [(-2)\mathbf{3}+\mathbf{4}]}} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 2 & 0 & 2 & 0 & 0 \\ 3 & 0 & 2 & 0 & 0 \\ \hline 1 & -2 & -2 & 2 & 4 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -2 & -1 \\ 0 & 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\left\{ \boldsymbol{x} \in \mathbb{R}^4 \;\middle|\; \text{exists } \begin{pmatrix} a \\ b \end{pmatrix} \in \mathbb{R}^2 \text{ such that } \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \\ -1 \\ 0 \end{pmatrix} + a \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + b \begin{pmatrix} 2 \\ 0 \\ -2 \\ 1 \end{pmatrix} \right\}$$

10 Complete algorithm to solve the linear system $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$

Apply elimination to solve $\left[\, \mathbf{A} \;\middle|\; -\boldsymbol{b} \,\right] \begin{pmatrix} \boldsymbol{x} \\ 1 \end{pmatrix} = \mathbf{0}$

$$\left[\begin{array}{c|c} \mathbf{A} & \text{-}\boldsymbol{b} \\ \hline \mathbf{I} & \mathbf{0} \\ \hline 0\cdots0 & 1 \end{array}\right] \xrightarrow{\text{Elimination}} \left[\begin{array}{c|c} \mathbf{K} & \boldsymbol{c} \\ \hline \mathbf{E} & \boldsymbol{x}_p \\ \hline 0\cdots0 & 1 \end{array}\right], \quad \text{where} \quad \mathbf{K} = \mathbf{AE}.$$

- If $\boldsymbol{c} \neq \mathbf{0}$, the system $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$ is not solvable.
- If $\boldsymbol{c} = \mathbf{0}$ then $\boldsymbol{b} \in \mathcal{C}(\mathbf{A})$ and the set of solutions is

$$\left\{ \boldsymbol{x} \in \mathbb{R}^n \;\middle|\; \text{exists } \boldsymbol{y} \in \mathcal{N}(\mathbf{A}) \text{ such that } \boldsymbol{x} = \boldsymbol{x}_p + \boldsymbol{y} \right\}.$$

If $\mathcal{N}(\mathbf{A}) = \{\mathbf{0}\}$, then $\boldsymbol{x}_p$ is the unique solution.

11 Rouché-Frobenius theorem

Sist. $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$	$r = m = n$	$r = n < m$	$r = m < n$	$r < m;\quad r < n$
solutions				

$$\left[\begin{array}{c|c} \mathbf{A} & \text{-}\boldsymbol{b} \\ \hline \mathbf{I} & \mathbf{0} \\ \hline \mathbf{0} & 1 \end{array}\right] \xrightarrow{\boldsymbol{\tau}_1\cdots\boldsymbol{\tau}_k} \left[\begin{array}{c|c} \mathbf{R} & \text{-}\boldsymbol{b} \\ \hline \mathbf{E} & \mathbf{0} \\ \hline \mathbf{0} & 1 \end{array}\right] = \left[\begin{array}{ccccccc|c} \mathbf{1} & 0 & \cdots & 0 & \cdots & 0 & 0 & \text{-}b_1 \\ 0 & \mathbf{1} & \cdots & 0 & \cdots & 0 & 0 & \text{-}b_2 \\ \vdots & \vdots & & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \mathbf{1} & \cdots & 0 & 0 & \text{-}b_h \\ \vdots & \vdots & & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & \cdots & :\mathbf{1} & 0\mathbf{1} & :\text{-}b_k \\ \vdots & \vdots & & \vdots & & \vdots & \vdots & \vdots \\ 0: & 0: & \cdots & 0: & \cdots & 0:\mathbf{1} & :\mathbf{1}0 & \text{-}b_m \\ \hline \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \hline 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & 1 \end{array}\right]$$

where **A** (with order $m \times n$) has rank r; and where "$\mathbf{1}$" are pivots.

Questions of the Lecture 8

(L-8) Question 1. Which of these rules give a correct definition of the rank of **A**?

(a) The number of non-zero columns in **R** (reduced column echelon form).
(b) The number of columns minus the total number of rows.
(c) The number of columns minus the number of free columns.
(d) The number of ones in **R**.

(Strang, 2006, exercise 12 from section 2.2.)

(L-8) Question 2. Find the complete solution (also called the *general solution*) to

$$\begin{bmatrix} 1 & 3 & 1 & 2 \\ 2 & 6 & 4 & 8 \\ 0 & 0 & 2 & 4 \end{bmatrix} \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix}$$

(Strang, 2003, exercise 4 from section 3.4.)

(L-8) QUESTION 3. Find the complete solution (also called the *general solution*) to

$$\begin{bmatrix}1 & 2 & -1 & -2 & 1\\ 1 & 2 & 0 & 0 & 3\\ 2 & 4 & 1 & 2 & 9\end{bmatrix}\begin{pmatrix}x_1\\x_2\\x_3\\x_4\\x_5\end{pmatrix}=\begin{pmatrix}0\\-1\\-4\end{pmatrix}$$

(L-8) QUESTION 4. Resuelva el siguiente sistema de ecuaciones

$$\begin{aligned} x_1+x_3+x_5 &=1\\ x_2+x_4 &=1\\ x_1+x_2+x_3+x_4 &=2\\ x_3+x_4 &=2 \end{aligned}$$

(L-8) QUESTION 5.
Consider

$$\mathbf{A}=\begin{bmatrix}0 & 1 & 0 & 3\\ 0 & 2 & 0 & 6\end{bmatrix}; \qquad \boldsymbol{b}=\begin{pmatrix}b_1\\b_2\end{pmatrix}.$$

(a) Find the column echelon form
(b) Find the free variables
(c) Find the special solutions:
(d) $\mathbf{A}\boldsymbol{x}=\boldsymbol{b}$ is consistent (has a solution) when $\boldsymbol{b}$ satisfies $b_2 =$ ____.
(e) Find the complete solution to the system when b_2 satisfies the consistency condition.

(Strang, 2006, exercise 3 from section 2.2.)

(L-8) QUESTION 6. Carry out the same steps as in the previous problem to find the complete solution of $\mathbf{A}\boldsymbol{x}=\boldsymbol{b}$.

$$\mathbf{A}=\begin{bmatrix}0 & 0\\ 1 & 2\\ 0 & 0\\ 3 & 6\end{bmatrix}; \quad \boldsymbol{b}=\begin{pmatrix}b_1\\b_2\\b_3\\b_4\end{pmatrix}.$$

(Strang, 2006, exercise 4 from section 2.2.)

(L-8) QUESTION 7. Describe the set of attainable right-hand sides $\boldsymbol{b}$? (the column space $\mathcal{C}(\mathbf{A})$) for

$$\begin{bmatrix}1 & 0\\ 0 & 1\\ 2 & 3\end{bmatrix}\begin{pmatrix}u\\v\end{pmatrix}=\begin{pmatrix}b_1\\b_2\\b_3\end{pmatrix},$$

by finding the constraints on $\boldsymbol{b}$ (after elimination). What is the rank? give a particular solution to the system?
(Strang, 2006, exercise 6 from section 2.2.)

(L-8) QUESTION 8. Suppose a paint company paints automobiles, trains and planes.
Each automobile takes 10 man hours to prepare, 30 man hours to paint, and 12 man hours to add finishing touches (the painters are quite meticulous).
Each train takes 20 man hours to prepare, 75 man hours to paint, and 36 man hours to add finishing touches.
Each plane takes 40 man hours to prepare, 135 man hours to paint, and 64 man hours to add finishing touches.
If the paint company decides to use 760 man hours towards preparation, 2595 man hours towards painting, and 1224 man hours towards finishing touches each week, how many planes, trains and automobiles do they paint each week?

(L-8) QUESTION 9. Para el sistema $\mathbf{A}\boldsymbol{x}=\boldsymbol{b}$ dado por

$$\begin{bmatrix}1 & 0 & 4\\ 2 & 1 & 10\\ 3 & 1 & c\end{bmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}6\\14\\20\end{pmatrix}$$

(a) Encuentre el valor de c que hace a la matriz $\mathbf{A}$ no invertible. Use dicho valor en los apartados siguientes.
(b) Encuentre la solución completa al sistema $\mathbf{A}\boldsymbol{x}=\boldsymbol{b}$.
(c) Describa el sistema de ecuaciones mediante la visión por columnas (columnas de $\mathbf{A}$ y el vector $\boldsymbol{b}$), o bien mediante la visión por filas (las tres ecuaciones del sistema).

(L-8) QUESTION 10. Construct a matrix whose column space contains $(1,1,5)$ and $(0,3,1)$ and whose nullspace contains $(1,1,2)$.
(Strang, 2006, exercise 62 from section 2.2.)

(L-8) QUESTION 11. For which vectors $\boldsymbol{b}$ do these systems have a solution?

(a) $\begin{bmatrix}1 & 1 & 1\\ 0 & 1 & 1\\ 0 & 0 & 1\end{bmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=\begin{pmatrix}b_1\\b_2\\b_3\end{pmatrix}$ (b) $\begin{bmatrix}1 & 1 & 1\\ 0 & 1 & 1\\ 0 & 0 & 0\end{bmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=\begin{pmatrix}b_1\\b_2\\b_3\end{pmatrix}$

(Strang, 2006, exercise 24 from section 2.1.)

(L-8) QUESTION 12. Under what conditions on b_1 and b_2 (if any) does $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$ have a solution?

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 0 & 3 \\ 2 & 4 & 0 & 7 \end{bmatrix}, \qquad \boldsymbol{b} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}.$$

Find two vectors in the nullspace of $\mathbf{A}$, and the complete solution to $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$. (Strang, 2006, exercise 8 from section 2.2.)

(L-8) QUESTION 13. Sea la matriz

$$\underset{3\times 4}{\mathbf{B}} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & -0 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & -0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(a) Sin realizar la multiplicación, diga una base de $\mathcal{N}\left(\mathbf{B}\right)$, y el rango de $\mathbf{B}$. Explique su respuesta.

(b) ¿Cuál es la solución completa a $\mathbf{B}\boldsymbol{x} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$?

(L-8) QUESTION 14. For which right-hand sides (find a condition on $\boldsymbol{b}$) are these systems solvable?

(a)

$$\begin{bmatrix} 1 & 4 & 2 \\ 2 & 8 & 4 \\ -1 & -4 & -2 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

Is the column space $\mathcal{C}\left(\mathbf{A}\right)$ the whole 3 dimensional space $\mathbb{R}^3$? , or is it only a plane? a line? a point?

(b)

$$\begin{bmatrix} 1 & 4 \\ 2 & 9 \\ -1 & -4 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

Is the column space $\mathcal{C}\left(\mathbf{A}\right)$ the whole 3 dimensional space $\mathbb{R}^3$? , or is it only a plane? a line? a point? *Based on (Strang, 2006, exercise 22 from section 2.1.)*

(L-8) QUESTION 15. The complete solution to $\mathbf{A}\boldsymbol{x} = \boldsymbol{b} \in \mathbb{R}^m$ is:

$$\left\{ \boldsymbol{x} \in \mathbb{R}^3 \;\middle|\; \exists\, c_1, c_2 \in \mathbb{R} \text{ such that } \boldsymbol{x} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + c_1 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

What is $\mathbf{A}$?

(L-8) QUESTION 16. Mike, Shai, and Tara all decide that they are unhappy with the color scheme at Math II classroom, and they do something about it. They go down to the paint store, and each buy some paint. Mike buys one gallon of red paint, six gallons of blue paint, and one gallon of yellow paint. He spends 44 euros. Shai, on the other hand, buys no red paint, two gallons of blue paint and three gallons of yellow paint. He spends 24 euros. Tara finally buys one gallon of red paint and five gallons of blue paint, and spends 33 euros.

(a) How much does each color of paint cost?
(b) What is wrong with your answer to the previous problem?
(c) When Mike, Shai and Tara compare receipts, they realize that one of them was charged 4 too little. Who was it?
(d) Tras intentar dar respuesta a la pregunta anterior, se habrá dado cuenta de que es un tanto “trabajoso” dar con el resultado. Intente lo siguiente: genere la matriz ampliada $\left[\mathbf{A}|\boldsymbol{a} \quad \boldsymbol{b} \quad \boldsymbol{c}\right]$ donde $\mathbf{A}$ es la matriz de coeficientes del sistema de ecuaciones, y $\boldsymbol{a}$ es el vector de precios suponiendo que a Ana deberían haberle cobrado 4 euros más (es decir 48 en lugar de 44), $\boldsymbol{b}$ el vector de precios suponiendo que sólo a Belén deberían haberle cobrado 4 euros más, y $\boldsymbol{c}$ lo mismo para Carlos. Calcule la forma escalonada reducida de la matriz ampliada. A la vista de lo obtenido ¿cuanto vale cada bote de pintura? y ¿a quien han cobrado 4 euros de menos?

(L-8) QUESTION 17. Suponga que el sistema de ecuaciones $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$ es consistente (que tiene solución), donde $\underset{m\times n}{\mathbf{A}}$ y $\boldsymbol{x} = (x_1, ..., x_n)$. **Demuestre** las siguientes afirmaciones:

(a) $\boldsymbol{b} \in \mathcal{C}\left(\mathbf{A}\right)$.
(b) Si $\boldsymbol{x}_0$ es una solución particular del sistema, entonces cualquier vector de la forma $\boldsymbol{x}_0 + \boldsymbol{z}$, donde $\boldsymbol{z} \in \mathcal{N}\left(\mathbf{A}\right)$, es también solución del sistema.
(c) Demuestre que si hay dependencia lineal entre las columnas de $\mathbf{A}$, entonces hay más de una solución.

(L-8) QUESTION 18. Solve the following system of equations using Gaussian elimination.

$$\begin{cases} 3x + y + z = 6 \\ x - y - z = -2 \\ 4y + z = 3 \end{cases}$$

(L-8) QUESTION 19. Write these ancient problems in a 2 by 2 matrix form $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$, and solve them:

(a) X is twice as old as Y and their ages add to 39.
(b) $(x, y,) = (2, 5,)$ and $(x, y,) = (3, 7,)$ lie on the line $y = mx + c$. Find m and c.

(Strang, 1988, exercise 32 from section 1.4.)

(L-8) QUESTION 20. The parabola $y = a + bx + cx^2$ goes through the points $(x, y,) = (1, 4,)$, $(2, 8,)$ and $(3, 14,)$. Find and solve a matrix equation for the unknowns $\boldsymbol{x} = \left(a, \quad b, \quad c,\right)$.

(Strang, 1988, exercise 33 from section 1.4.)

(L-8) QUESTION 21. Explain why the system

$$\begin{cases} u + v + w = 2 \\ u + 2v + 3w = 1 \\ v + 2w = 0 \end{cases}$$

is singular and has no solution.
What value should replace the last zero on the right hand side, to allow the the equations to have solutions— and what is one of the solutions?
(Strang, 1988, exercise 8 from section 1.2.)

(L-8) QUESTION 22. Choose a coefficient b that makes this system singular. Then choose a value for g that makes it solvable. Find two solutions in that singular case.

$$\begin{cases} 2x + by &= 16 \\ 4x + 8y &= g \end{cases}$$

Based on (Strang, 2003, exercise 6 from section 2.2.)

L-6 L-7 L-8 L-9 L-10 L-R

(L-8) QUESTION 23. Solve the following nonsingular triangular system. Show that your solution gives the linear combination of the columns that equals the column of the right $\boldsymbol{b} = (b_1, \quad b_2, \quad b_3,)$.

$$\begin{aligned} u - v + w &= b_1 \\ v + w &= b_2 \\ w &= b_3. \end{aligned}$$

Check your answer multiplying **A** by your solution vector.
(Strang, 1988, exercise 2 from section 1.2.)

(L-8) QUESTION 24. Find **A** and **B** with the given property or explain why you can't.

(a) The only solution to $\mathbf{A}\boldsymbol{x} = \begin{pmatrix}1\\2\\3\end{pmatrix}$ is $\boldsymbol{x} = \begin{pmatrix}0\\1\end{pmatrix}$.

(b) The only solution to $\mathbf{B}\boldsymbol{x} = \begin{pmatrix}0\\1\end{pmatrix}$ is $\boldsymbol{x} = \begin{pmatrix}1\\2\\3\end{pmatrix}$.

(Strang, 2006, exercise 49 from section 2.2.)

L-6 L-7 L-8 L-9 L-10 L-R

(L-8) QUESTION 25. The complete solution to $\mathbf{A}x = \begin{pmatrix}1\\3\end{pmatrix}$ is $\boldsymbol{x} = \begin{pmatrix}1\\0\end{pmatrix} + c\begin{pmatrix}0\\1\end{pmatrix}$. Find **A**.
(Strang, 2006, exercise 50 from section 2.2.)

(L-8) QUESTION 26. Suppose the fifth column of **L** has no pivot. Then x_5 is a ____________ variable. The zero vector (is) (is not) the only solution to $\mathbf{A}\boldsymbol{x} = \mathbf{0}$. If $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$ has a solution, then it has____________ solutions.
(Strang, 2006, exercise 40 from section 2.2.)

(L-8) QUESTION 27. Consider a linear system of algebraic equation $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$. Here the matrix **A** has three rows and four columns.

(a) Does such a linear system always have at least one solution? If not provide an example for which no solution exists.
(b) Can such a linear system have a unique solution? If so, provide and example of a problem with this property.
(c) Formulate, if possible, necessary and sufficient conditions on **A** and $\boldsymbol{b}$ which guarantee that at least one solution exists.
(d) Formulate, if possible, necessary and sufficient conditions on **A** which guarantee that at least one solution exists for any choice of $\boldsymbol{b}$.

L-6 L-7 L-8 L-9 L-10 L-R

(L-8) QUESTION 28. By performing column eliminations on the 4×7 matrix **A**

$$\mathbf{A} = \begin{bmatrix} 1 & -1 & 0 & 2 & 0 & -1 & 0 \\ 2 & -2 & 1 & 5 & 0 & -1 & 0 \\ -3 & 3 & -1 & -7 & 2 & 2 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 1 \end{bmatrix}$$

we got the following matrix $\mathbf{B} = \mathbf{A}_{\tau_1 \cdots \tau_k}$:

$$\mathbf{B} = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}; \quad \text{where } \mathbf{I}_{\tau_1 \cdots \tau_k} = \begin{bmatrix} 2 & 1 & 0 & -2 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ -4 & 0 & 2 & -1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix}.$$

(a) What is the rank of **A**? Find the complete solution to $\mathbf{A}\boldsymbol{x} = \mathbf{0}$.
(b) Write, if it is possible, the general solution as a function of x_2, x_4, and x_6.
(c) Is it possible to find a vector $\boldsymbol{b}$ in $\mathbb{R}^4$ that is not in the column space of **A** ($\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$ has no solution)? If it is, give an example.
(d) Give a vector $\boldsymbol{b}$ such that the vector $\boldsymbol{x} = \mathbf{I}_{|1}$ is a solution to the system $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$.
(e) If $\boldsymbol{b}$ is the sum of columns of **A**, find, if it is possible, the full solution to $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$.

A modified version of MIT Course 18.06 Quiz 1, October 4, 2004

(L-8) QUESTION 29. $\underset{m\times n}{\mathbf{A}}$ es una matriz de rango r. Suponga que $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$ no tiene solución para algunos vectores $\boldsymbol{b}$, pero infinitas soluciones para otros vectores $\boldsymbol{b}$.

(a) Decida si el espacio nulo $\mathcal{N}(\mathbf{A})$ contiene sólo el vector cero, y explique porqué.
(b) Decida si el espacio columna $\mathcal{C}(\mathbf{A})$ es todo $\mathbb{R}^m$ y explique porqué.
(c) Para esta matriz $\mathbf{A}$, encuentre las relaciones entre los números r, m; y entre r y n.

(d) ¿Puede existir un lado derecho $\boldsymbol{b}$ para el que $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$ tenga una y sólo una solución? ¿Porqué es posible o porqué no?

(L-8) QUESTION 30. Sea la matriz

$$\mathbf{A} = \begin{bmatrix} 2 & 3 & 1 & -1 \\ 6 & 9 & 3 & -2 \end{bmatrix}$$

(a) Encuentre un conjunto de soluciones del sistema $\mathbf{A}\boldsymbol{x} = \mathbf{0}$ y describa con él el espacio nulo de $\mathbf{A}$.
(b) Encuentre la solución completa— es decir todas las soluciones (x_1, x_2, x_3, x_4) — de

$$\mathbf{A}\boldsymbol{x} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

(c) Cuando una matriz $\underset{m\times n}{\mathbf{A}}$ tiene rango $r = m$ ¿para qué vectores $\boldsymbol{b}$ el sistema $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$ puede resolverse? ¿Cuantas soluciones especiales tiene $\mathbf{A}\boldsymbol{x} = \mathbf{0}$ (dimensión del espacio nulo)?

(L-8) QUESTION 31. consider the system of linear equations,

$$\begin{cases} x + y + 2z = 1 \\ 2x + 2y - z = 1 \\ y + cz = 2 \end{cases}$$

For which number c the system has no solution? Only one? and infinite solutions?

(L-8) QUESTION 32. Consider the following system of linear equations

$$\begin{cases} x - y + 2z = 1 \\ 2x - 3y + mz = 3 \\ -x + 2y + 3z = 2m \end{cases}$$

(a) Show that the system has solution for any value m
(b) Find the solution when $m = -1$.
(c) Is the set of solutions to the system in the last question $(m = -1)$ a line in $\mathbb{R}^3$? Is there any m such as the set of solutions to the system is a plane in $\mathbb{R}^3$?... and a point in $\mathbb{R}^3$?
(d) Find the solution to the system when $m = 1$.

(L-8) QUESTION 33. Which descriptions are correct? The solutions $\boldsymbol{x}$ of

$$\mathbf{A}\boldsymbol{x} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 2 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

form:

(a) A plane
(b) A line
(c) A point
(d) A subspace
(e) The nullspace of $\mathbf{A}$.
(f) The column space of $\mathbf{A}$.

(Strang, 2006, exercise 8 from section 2.1.)

(L-8) QUESTION 34. Consider the equation $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$

$$\begin{bmatrix} 1 & 0 \\ 4 & 1 \\ 2 & -1 \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

For which $\boldsymbol{b}$ are there solutions?
Based in *MIT Course 18.06 Quiz 1, Fall 2008*

(L-8) QUESTION 35. En un teatro de barrio, tres grupos están haciendo cola. Hay cuatro tipos de tarifas; tercera edad (t), adulto (a), infantil (i) y tarifa con descuento para empleados del teatro y familiares (d).
El primer grupo compra tres entradas de adulto y tres infantiles por 39 euros.
El segundo grupo compra tres entradas de adulto y cuatro de la tercera edad por 44 euros
El tercer grupo compra dos entradas con descuento y dos entradas infantiles por 22 euros

(a) Si intenta descubrir el precio de cada entrada ¿cuantas soluciones puede encontrar? Ninguna, una, o infinitas
(b) Si las entradas de la tercera edad valen lo mismo que las infantiles. ¿Cuánto vale cada tipo de entrada?

(L-8) QUESTION 36. Consider the following system of linear equations

$$\begin{cases} x_1 + 2x_2 - x_3 + x_4 = -1 \\ -x_1 - 2x_2 + 3x_3 + 5x_4 = -5 \\ -x_1 - 2x_2 - x_3 - 7x_4 = 7 \end{cases}$$

(a) (0.5 pts) What is the rank of the coeficient matrix?
(b) (1.5 pts) Find all solutions to the system of linear equations
(c) (0.5 pts) Describe the geometric shape of the collection of all solutions to the above equations considered as a subset of $\mathbb{R}^4$.

(L-8) QUESTION 37. Consider the following linear system $\mathbf{A}\boldsymbol{x} = \mathbf{0}$ where

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 1 & 2 & 3 \\ a & 1 & 1 & 2 \end{bmatrix}.$$

(a) (0.5^{pts}) Find the values of a such as the set of solutions of the linear system is a line.
(b) (0.5^{pts}) Find the values of a such as the set of solutions of the linear system is a plane?

(L-8) QUESTION 38. Find the complete solution to the system

$$\begin{bmatrix} 1 & 3 & 2 & 4 & -3 \\ 2 & 6 & 0 & -1 & -2 \\ 0 & 0 & 6 & 2 & -1 \\ 1 & 3 & -1 & 4 & 2 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} -7 \\ 0 \\ 12 \\ -6 \end{pmatrix}$$

(L-8) QUESTION 39. Sea la matriz $\underset{3\times 4}{\mathbf{A}}$ y el vector columna $\boldsymbol{b}$ de $\mathbb{R}^3$:

$$\mathbf{A} = \begin{bmatrix} 1 & 3 & 2 & 2 \\ 2 & 7 & 6 & 8 \\ 3 & 9 & 6 & 7 \end{bmatrix}; \qquad \boldsymbol{b} = \begin{pmatrix} 2 \\ 7 \\ 7 \end{pmatrix}$$

(a) Encuentre todas la soluciones al sistema $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$ (si es que existen soluciones). Describa el conjunto de soluciones geométricamente. ¿Es dicho conjunto un sub-espacio vectorial?
(b) ¿Quién es el espacio columna $\mathcal{C}(\mathbf{A})$? Cambie el 7 de la esquina inferior derecha por un número que conduzca a un espacio columna más pequeño de la nueva matriz (digamos $\mathbf{M}$). Dicho número es ____.
(c) Encuentre un lado derecho $\boldsymbol{b}$ tal que, para la nueva matriz, el sistema $\mathbf{M}\boldsymbol{x} = \boldsymbol{b}$ tenga solución; y otro lado derecho $\boldsymbol{b}$ tal que $\mathbf{M}\boldsymbol{x} = \boldsymbol{b}$ no tenga solución.

1 Highlights of **Lesson 9**

Highlights of Lesson 9

- Linear *independence*
- vectors *spanning* a space
- **BASIS** and dimension

2 Homogeneous equations: our starting point

Suppose I have a matrix $\underset{m\times n}{\mathbf{A}}$ with $m < n$ and I look at $\mathbf{A}\boldsymbol{x} = \mathbf{0}$.
(more unknowns than equations ($m < n$), *free columns*)

Then, there's something in $\mathcal{N}(\mathbf{A})$, other than just the zero vector.

There are non-trivial linear combinations $\mathbf{A}\boldsymbol{x}$ that give $\mathbf{0}$

3 Linear independence

Vectors $\vec{v}_1, \ldots, \vec{v}_n$ are (*linearly*) independent if:

the only linear combination that is equal to $\vec{0}$ is

$$(\vec{v}_1)0 + (\vec{v}_2)0 + \cdots + (\vec{v}_n)0$$

that is

$(\vec{v}_1)p_1 + \cdots + (\vec{v}_n)p_n = \vec{0}$ only happens when all p_i are zero

$[\vec{v}_1; \ldots \vec{v}_n;]\boldsymbol{p} = \vec{0}$ if and only if $\boldsymbol{p} = \mathbf{0}$

4 linear independence: examples in $\mathbb{R}^2$

Can you find numbers a and b such that $a\boldsymbol{v} + b\boldsymbol{w} = \mathbf{0}$?

- $\boldsymbol{v}$ and $\boldsymbol{w} = 2\boldsymbol{v}$
- $\boldsymbol{v}$ and $\boldsymbol{w} = \mathbf{0}$
- 2 non-aligned vectors
- 3 vectors in $\mathbb{R}^2$

5 linear independence and rank of a matrix

Columns of $\underset{m\times n}{\mathbf{A}}$ are:

- independent:

  If the null space $\mathcal{N}(\mathbf{A})$ *is*
- dependent if:

  $\mathbf{A}\boldsymbol{c} = \mathbf{0}$ *for some vector* $\boldsymbol{c} \neq \mathbf{0}$.
- independent if: $\operatorname{rg}(\mathbf{A})$
- dependent if: $\operatorname{rg}(\mathbf{A})$

6 Space spanned by a system of vectors: Generating system

Generating system

The system $\mathsf{Z} = [\vec{z}_1; \ldots \vec{z}_j;]$ **spans** subspace $\mathcal{W}$ if their linear combinations fill $\mathcal{W}$

- $\mathcal{W}$ consists of all linear combinations of $\vec{z}_1, \ldots \vec{z}_j$.
- $\mathcal{W}$ is the smallest subspace that contains Z.

$$\mathcal{W} = \mathcal{L}\Big([\vec{z}_1; \ldots \vec{z}_j;]\Big).$$

Example

- The column space:

$$\mathcal{C}(\mathbf{A}) = \{\boldsymbol{b} \mid \exists \boldsymbol{x} \text{ such that } \boldsymbol{b} = \mathbf{A}\boldsymbol{x}\} = \mathcal{L}\Big(\text{columns of } \mathbf{A}\Big).$$

- The null space:

$$\mathcal{N}(\mathbf{A}) = \{\boldsymbol{x} \mid \mathbf{A}\boldsymbol{x} = \mathbf{0}\} = \mathcal{L}\Big(\text{special solutions to } \mathbf{A}\boldsymbol{x} = \mathbf{0}\Big).$$

7 A Basis for a Vector Space

Basis for a subspace $\mathcal{W}$

is a system of vectors $[\vec{z}_1; \dots \vec{z}_d;]$ such that;

1. span the subspace $\mathcal{W}$
2. are *linearly independent*

examples

$\mathbb{R}^3$:

$[\boldsymbol{a}_1; \dots \boldsymbol{a}_n;]$ is a basis of $\mathbb{R}^n$ if it is an invertible matrix

All bases for a given subspace $\mathcal{W}$ contain the same *number* of vectors

8 Dimension

All bases for a given subspace $\mathcal{S}$ contain the same *number* of vectors

The dimension of a space is that number

That number indicates the "size" of the space

9 Examples: $\mathcal{C}(\mathbf{A})$

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 2 & 3 & 1 \end{bmatrix};$$

- do the columns span $\mathcal{C}(\mathbf{A})$?
- are the columns a basis for $\mathcal{C}(\mathbf{A})$?
- What is the rank of $\mathbf{A}$?
- write down some bases for $\mathcal{C}(\mathbf{A})$

$\mathrm{rg}(\mathbf{A}) =$ num. pivots $=$ dimension of $\mathcal{C}(\mathbf{A})$

10 Examples: $\mathcal{N}(\mathbf{A})$

$$\underset{m\times n}{\mathbf{A}} = \begin{bmatrix} 1 & 2 & 3 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 2 & 3 & 1 \end{bmatrix}; \qquad \boldsymbol{v} = \begin{pmatrix} -1 \\ -1 \\ 1 \\ 0 \end{pmatrix}$$

- Is $\boldsymbol{v}$ in $\mathcal{N}(\mathbf{A})$?
- Does $\boldsymbol{v}$ span $\mathcal{N}(\mathbf{A})$?
- write down another vector of $\mathcal{N}(\mathbf{A})$ independent of $\boldsymbol{v}$.
- do $\boldsymbol{v}$ and $\boldsymbol{w}$ span $\mathcal{N}(\mathbf{A})$?
- are $\boldsymbol{v}$ and $\boldsymbol{w}$ a basis for $\mathcal{N}(\mathbf{A})$?
- What is the dimension of $\mathcal{N}(\mathbf{A})$?

$n - \mathrm{rg}(\mathbf{A}) =$ num. free variables $= \dim \mathcal{N}(\mathbf{A})$

Questions of the Lecture 9

(L-9) QUESTION 1. Decide whether or not the following vectors are linearly independent, by solving $c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2 + c_3\boldsymbol{v}_3 + c_4\boldsymbol{v}_4 = \mathbf{0}$:

$$\boldsymbol{v}_1 = \begin{pmatrix}1\\1\\0\\0\end{pmatrix}, \quad \boldsymbol{v}_2 = \begin{pmatrix}1\\0\\1\\0\end{pmatrix}, \quad \boldsymbol{v}_3 = \begin{pmatrix}0\\0\\1\\1\end{pmatrix}, \quad \boldsymbol{v}_4 = \begin{pmatrix}0\\1\\0\\1\end{pmatrix}.$$

Decide also if they span $\mathbb{R}^4$, by trying to solve $c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2 + c_3\boldsymbol{v}_3 + c_4\boldsymbol{v}_4 = (0, \quad 0, \quad 0, \quad 1,)$.
(Strang, 2006, exercise 16 from section 2.3.)

(L-9) QUESTION 2. Suppose $\boldsymbol{v}_1 \dots \boldsymbol{v}_6$ are six vectors in $\mathbb{R}^4$.

(a) Those vectors (do)(do not)(might not) span $\mathbb{R}^4$.
(b) Those vectors (are)(are not)(might be) linearly independent.
(c) If those vectors are the columns of $\mathbf{A}$, then $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$ (has) (does not have) (might not have) a solution.
(d) If those vectors are the columns of $\mathbf{A}$, then $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$ (has) (does not have) (might not have) a sole solution.

(Strang, 2006, exercise 22 from section 2.3.)

(L-9) QUESTION 3. Find **B** and **C** with the given property or explain why you can't.

(a) The complete solution to $\mathbf{B}\boldsymbol{x} = \begin{pmatrix}1\\2\\4\end{pmatrix}$ is $\boldsymbol{x} = \begin{pmatrix}1\\0\end{pmatrix}$. Find **B** or explain why you can't.

(b) The complete solution to $\mathbf{C}\boldsymbol{x} = \begin{pmatrix}5\\1\end{pmatrix}$ is $\boldsymbol{x} = \begin{pmatrix}1\\4\\-2\end{pmatrix}$. Find **C** or explain why you can't.

(L-9) QUESTION 4. Show that $\boldsymbol{v}_1$, $\boldsymbol{v}_2$, $\boldsymbol{v}_3$ are independent but $\boldsymbol{v}_1$, $\boldsymbol{v}_2$, $\boldsymbol{v}_3$, $\boldsymbol{v}_4$ are dependent:

$$\boldsymbol{v}_1 = \begin{pmatrix}1\\0\\0\end{pmatrix}; \quad \boldsymbol{v}_2 = \begin{pmatrix}1\\1\\0\end{pmatrix}; \quad \boldsymbol{v}_3 = \begin{pmatrix}1\\1\\1\end{pmatrix}; \quad \boldsymbol{v}_4 = \begin{pmatrix}2\\3\\4\end{pmatrix}.$$

Solve $\mathbf{A}\boldsymbol{c} = \mathbf{0}$ (where the $\boldsymbol{v}$'s go in the columns of $\mathbf{A}$).
(Strang, 2006, exercise 1 from section 2.3.)

(L-9) QUESTION 5. True or false?

If $\mathbf{A}^\intercal = 2\mathbf{A}$, then the rows of $\mathbf{A}$ are linearly dependent.

(L-9) QUESTION 6. Which of the following sets of vectors span $\mathbb{R}^3$?

(a) $(1, 2, 0,)$ and $(0, -1, 1,)$.
(b) $(1, 1, 0,)$, $(0, 1, -2,)$, and $(1, 3, 1,)$.
(c) $(-1, 2, 3,)$, $(2, 1, -1,)$, and $(4, 7, 3,)$.
(d) $(1, 0, 2,)$, $(0, 1, 0,)$, $(-1, 3, 0,)$, and $(1, -4, 1,)$.

(L-9) QUESTION 7. Which of the following systems of vectors are linearly independent? In case of linear dependence, write one of the vectors as a linear combination of the others.

(a) $(-1, 2, 3,)$, $(2, 1, -1,)$, and $(4, 7, 3,)$ in $\mathbb{R}^3$.
(b) $(1, 2, 0,)$ and $(0, -1, 1,)$ in $\mathbb{R}^3$.
(c) $(1, 2,)$, $(2, 3,)$, and $(8, -2,)$ in $\mathbb{R}^2$.
(d) $t^2 + 2t + 1, \quad t^3 - t^2, \quad t^3 + 1,$ and $t^3 + t + 1$ in P_3.

(L-9) QUESTION 8. Suppose the only solution to $\underset{m\times n}{\mathbf{A}}\ \boldsymbol{x} = \mathbf{0}$ (m equations in n unknowns) is $\boldsymbol{x} = \mathbf{0}$. What is the rank and why? The columns of A are linearly __________.
(Strang, 2006, exercise 8 from section 2.4.)

(L-9) QUESTION 9. **[Important]** If **A** has order 4×6, prove that the columns of **A** are linearly dependent.

(L-9) QUESTION 10. **A** is such that $\mathcal{N}(\mathbf{A}) = \mathcal{L}\left(\left[\begin{pmatrix}1\\2\\-1\\3\end{pmatrix}; \begin{pmatrix}0\\1\\1\\4\end{pmatrix}; \begin{pmatrix}-1\\-1\\3\\1\end{pmatrix};\right]\right)$.

(a) Find a matrix **B** such that its column space $\mathcal{C}(\mathbf{B}) = \mathcal{N}(\mathbf{A})$. [Thus, any vector $\boldsymbol{y} \in \mathcal{N}(A)$ satisfies $\mathbf{B}\boldsymbol{u} = \boldsymbol{y}$ for some $\boldsymbol{u}$.]
(b) Give a different possible answer to (a): another **B** with $\mathcal{C}(\mathbf{B}) = \mathcal{N}(\mathbf{A})$.
(c) For some vector $\boldsymbol{b}$, you are told that a particular solution to $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$ is

$$\boldsymbol{x}_p = (1, \quad 2, \quad 3, \quad 4,)$$

Now, your classmate Zarkon tells you that a second solution is:

$$\boldsymbol{x}_Z = (1, \quad 1, \quad 3, \quad 0,)$$

while your other classmate Hastur tells you "No, Zarkon's solution can't be right, but here's a second solution that is correct:"

$$\boldsymbol{x}_H = (1, \quad 1, \quad 3, \quad 1,)$$

Is Zarkon's solution correct, or Hastur's solution, or are both correct? (Hint: what should be true of $\boldsymbol{x}$ - $\boldsymbol{x}_p$ if $\boldsymbol{x}$ is a valid solution?)

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(L-9) QUESTION 11. Consider the matrix

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & -1 & 0 & 0 \\ 1 & 2 & 0 & 2 & 2 \\ 1 & 2 & -1 & 0 & 0 \\ 2 & 4 & 0 & 4 & 4 \end{bmatrix}$$

(a) Find a basis of the column space $\mathcal{C}(\mathbf{A})$.
(b) Find a basis of the nullspace $\mathcal{N}(\mathbf{A})$.
(c) Find linear conditions on a, b, c, d that guarantee that the system $\mathbf{A}\boldsymbol{x} = (a, b, c, d,)$ has a solution.
(d) Find the complete solution for the system $\mathbf{A}\boldsymbol{x} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 2 \end{pmatrix}$.

MIT Course 18.06 Quiz 1, March 5, 2007

(L-9) QUESTION 12. Si a una matriz $\mathbf{A}$ se le "añade" una nueva columna extra $\boldsymbol{b}$, entonces el espacio columna se vuelve más grande, a no ser que ____________. Proporcione un ejemplo en el que espacio columna se haga más grande, y uno en el que no. ¿Por qué $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$ es resoluble cuando el espacio columna no crece al añadir $\boldsymbol{b}$?

(L-9) QUESTION 13. If the 9 by 12 system $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$ is solvable for every $\boldsymbol{b}$, then $\mathcal{C}(\mathbf{A}) =$ ____
(Strang, 2006, exercise 30 from section 2.1.)

(L-9) QUESTION 14. **[Importante]**[1] Suponga que el sistema $\left[\boldsymbol{v}_1; \ \dots \ \boldsymbol{v}_n\right]$ de vectores de $\mathbb{R}^m$ genera el subespacio $\mathcal{V}$, y suponga que $\boldsymbol{v}_n$ es una combinación lineal de los vectores $\boldsymbol{v}_1, \dots \boldsymbol{v}_{n-1}$. Demuestre que el sistema $\left[\boldsymbol{v}_1; \ \dots \ \boldsymbol{v}_{n-1}\right]$ también genera el subespacio $\mathcal{V}$.

(L-9) QUESTION 15.

(a) Find the general (complete) solution to this equation

$$\begin{bmatrix} 1 & 1 & 2 \\ 1 & 1 & 2 \\ 2 & 2 & 2 \end{bmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 4 \end{pmatrix}$$

(b) Find a basis for the column of the 3 by 9 block matrix $\left[\mathbf{A}; \quad 2\mathbf{A}; \quad \mathbf{A}^2;\right]$.

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[1] *pista:* Piense si el espacio $\mathcal{V}$ se puede expresar como el espacio columna de una matriz $\mathbf{V}$ cuyas columnas son los vectores $\boldsymbol{v}_1, \dots, \boldsymbol{v}_n$. Una vez expresado de esa manera, recuerde que las operaciones entre columnas no alteran el espacio columna de la matriz. Por último, transforme $\mathbf{V}$ de manera que transforme una de las columnas en un vector de ceros.

(L-9) QUESTION 16. ¿Cuáles de los siguientes vectores generan el espacio de polinomios de, a lo sumo, grado 4; es decir, el conjunto de polinomios $P_3 = \{at^3 + bt^2 + ct + d\}$?

(a) $t + 1, \ t^2 - t, \ \text{y} \ t^3$.
(b) $t^3 + t \ \text{y} \ t^2 + 1$.
(c) $t^2 + t + 1, \ t + 1, \ 1, \ \text{y} \ t^3$.
(d) $t^3 + t^2, \ t^2 - t, \ 2t + 4, \ \text{y} \ t^3 + 2t^2 + t + 4$.

(L-9) QUESTION 17. Considere los vectores $\boldsymbol{u}_1 = (1, 0, 1,)$ y $\boldsymbol{u}_2 = (1, -1, 1,)$.

(a) **Demuestre** que $\boldsymbol{u}_1$ y $\boldsymbol{u}_2$ son linealmente independientes.
(b) ¿Pertenece $\boldsymbol{v} = (2, 1, 2,)$ al espacio generado por $\{\boldsymbol{u}_1, \boldsymbol{u}_2\}$? Explique las razones de su respuesta.
(c) Encuentre una base de $\mathbb{R}^3$ que contenga a $\boldsymbol{u}_1$ y a $\boldsymbol{u}_2$. Explique su respuesta.

(L-9) QUESTION 18.

(a) ¿Son linealmente independientes los siguientes vectores? Explique su respuesta.

$$\boldsymbol{v}_1 = \begin{pmatrix} -2 \\ -1 \\ 3 \\ 4 \end{pmatrix}; \qquad \boldsymbol{v}_2 = \begin{pmatrix} -8 \\ 2 \\ -2 \\ 1 \end{pmatrix}$$

(b) ¿Son los siguientes vectores una base de $\mathbb{R}^4$? Explique su respuesta.

$$\boldsymbol{v}_1 = \begin{pmatrix} -2 \\ -1 \\ 3 \\ 4 \end{pmatrix}; \quad \boldsymbol{v}_2 = \begin{pmatrix} 8 \\ 2 \\ 2 \\ 1 \end{pmatrix}; \quad \boldsymbol{v}_3 = \begin{pmatrix} 10 \\ 1 \\ 1 \\ 6 \end{pmatrix}; \quad \boldsymbol{v}_4 = \begin{pmatrix} -2 \\ -1 \\ 3 \\ 4 \end{pmatrix}$$

(c) ¿Son los siguientes vectores una base del subespacio descrito por el plano tridimensional $x_1 + 2x_2 + 3x_3 + 6x_4 = 0$? Explique su respuesta.

$$\boldsymbol{v}_1 = \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix}; \quad \boldsymbol{v}_2 = \begin{pmatrix} -1 \\ -1 \\ 1 \\ 0 \end{pmatrix}; \quad \boldsymbol{v}_3 = \begin{pmatrix} -4 \\ -2 \\ 2 \\ 1 \end{pmatrix}$$

(d) Encuentre el valor de q para el que los siguientes vectores no generan $\mathbb{R}^3$.

$$\boldsymbol{v}_1 = \begin{pmatrix} 1 \\ 4 \\ 6 \end{pmatrix}; \quad \boldsymbol{v}_2 = \begin{pmatrix} 0 \\ 2 \\ 2 \end{pmatrix}; \quad \boldsymbol{v}_3 = \begin{pmatrix} -1 \\ 12 \\ 10 \end{pmatrix}; \quad \boldsymbol{v}_4 = \begin{pmatrix} q \\ 3 \\ 1 \end{pmatrix}$$

(L-9) QUESTION 19. Suponga que tiene 4 vectores columna $\boldsymbol{u}$, $\boldsymbol{v}$, $\boldsymbol{w}$ y $\boldsymbol{z}$ en el espacio tridimensional $\mathbb{R}^3$.

(a) Dé un ejemplo donde el espacio columna de $\mathbf{A}$ contenga $\boldsymbol{u}$, $\boldsymbol{v}$ y $\boldsymbol{w}$, pero no a $\boldsymbol{z}$. (escriba unos vectores $\boldsymbol{u}$, $\boldsymbol{v}$, $\boldsymbol{w}$ y $\boldsymbol{z}$; y una matriz $\mathbf{A}$ que cumplan lo anterior).

(b) ¿Cuáles son las dimensiones del espacio columna y del espacio nulo de su matriz ejemplo $\mathbf{A}$ del apartado anterior?

1 Highlights of Lesson 10

Highlights of Lesson 10

- The Four Fundamental Subspaces of a matrix $\mathbf{A}$
  - Column space $\mathcal{C}\left(\mathbf{A}\right)$
  - Nullspace $\mathcal{N}\left(\mathbf{A}\right)$
  - Row space $\mathcal{C}\left(\mathbf{A}^\intercal\right)$
  - Left nullspace $\mathcal{N}\left(\mathbf{A}^\intercal\right)$

2 The Four Fundamental Subspaces of a matrix A

Where are those subspaces if $\underset{m\times n}{\mathbf{A}}$?

- Column space $\mathcal{C}\left(\mathbf{A}\right)$
- Nullspace $\mathcal{N}\left(\mathbf{A}\right)$
- Row space
  Linear combinations of the rows
  Linear combinations of the columns of $\mathbf{A}^\intercal = \mathcal{C}\left(\mathbf{A}^\intercal\right)$
- Left nullspace of $\mathbf{A}$, $\quad\mathcal{N}\left(\mathbf{A}^\intercal\right)$

3 Bases for the 4 subspaces: row space

$$\left[\frac{\mathbf{A}}{\mathbf{I}}\right]=\left[\begin{array}{cccc}1&2&3&1\\1&1&2&1\\1&2&3&1\\\hline 1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{array}\right]\xrightarrow[\substack{[(-2)\mathbf{1}+\mathbf{2}]\\ [(-3)\mathbf{1}+\mathbf{3}]\\ [(-1)\mathbf{1}+\mathbf{4}]}]{\boldsymbol{\tau}}\left[\begin{array}{cccc}1&0&0&0\\1&-1&-1&0\\1&0&0&0\\\hline 1&-2&-3&-1\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{array}\right]\xrightarrow[\substack{[(-1)\mathbf{2}]\\ [(-1)\mathbf{2}+\mathbf{1}]\\ [(1)\mathbf{2}+\mathbf{3}]}]{\boldsymbol{\tau}}\left[\begin{array}{cccc}1&0&0&0\\0&1&0&0\\1&0&0&0\\\hline -1&2&-1&-1\\1&-1&-1&0\\0&0&1&0\\0&0&0&1\end{array}\right]=\left[\frac{\mathbf{R}}{\mathbf{E}}\right]$$

column operations preserve $\mathcal{C}\left(\mathbf{A}\right)$ (but not the row space)

$$\mathcal{C}\left(\mathbf{A}^\intercal\right)\neq\mathcal{C}\left(\mathbf{L}^\intercal\right)\neq\mathcal{C}\left(\mathbf{R}^\intercal\right);\quad (1,\ 2,\ 3,\ 1,)\in\mathcal{C}\left(\mathbf{A}^\intercal\right)\text{ but }\notin\mathcal{C}\left(\mathbf{R}^\intercal\right)$$

What's the dimension of the row space $\mathcal{C}\left(\mathbf{A}^\intercal\right)$?
What's a basis for the row space of $\mathbf{A}$?
A basis for the column space of $\mathbf{A}$?

4 Left null space: why that name?

$\mathcal{N}(\mathbf{A}^\intercal)$

$$(\mathbf{A}^\intercal)\boldsymbol{y} = \mathbf{0}$$

$$\begin{bmatrix} & \\ & \mathbf{A}^\intercal & \\ & \end{bmatrix}\begin{pmatrix} y_1 \\ \vdots \\ y_m \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$$

so...

$$\boldsymbol{y}\mathbf{A} = \mathbf{0}$$

$$(y_1, \quad \dots \quad y_m,)\begin{bmatrix} & \\ & \mathbf{A} & \\ & \end{bmatrix} = (0, \quad \dots \quad 0,)$$

5 Column elimination preserves the left null space

Let $\underset{n\times n}{\mathbf{E}} = \mathbf{I}_{\tau_1\cdots\tau_k}$ be invertible then

- If $\boldsymbol{x} \in \mathcal{N}(\mathbf{A}^\intercal)$

$$\boldsymbol{x}\mathbf{A} = \mathbf{0} \quad \text{and} \quad \boldsymbol{x}\mathbf{AE} = \mathbf{0E} = \mathbf{0} \quad \Rightarrow \boldsymbol{x} \in \mathcal{N}((\mathbf{AE})^\intercal);$$

- If $\boldsymbol{x} \in \mathcal{N}((\mathbf{AE})^\intercal)$

$$\boldsymbol{x}\mathbf{AE} = \mathbf{0} \quad \text{and} \quad \boldsymbol{x}\mathbf{A} = \mathbf{0E}^{-1} = \mathbf{0} \quad \Rightarrow \boldsymbol{x} \in \mathcal{N}(\mathbf{A}^\intercal).$$

Therefore,

$$\mathcal{N}(\mathbf{A}^\intercal) = \mathcal{N}((\mathbf{AE})^\intercal) = \mathcal{N}\left((\mathbf{A}_{\tau_1\cdots\tau_k})^\intercal\right).$$

6 Finding a basis of $\mathcal{N}(\mathbf{A}^\intercal)$ by column reduction

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 2 & 3 & 1 \end{bmatrix} \xrightarrow[{[(-1)\mathbf{1}+\mathbf{4}]}]{\substack{\boldsymbol{\tau} \\ [(-2)\mathbf{1}+\mathbf{2}] \\ [(-3)\mathbf{1}+\mathbf{3}]}} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & -1 & -1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \xrightarrow[{[(1)\mathbf{2}+\mathbf{3}]}]{\substack{\boldsymbol{\tau} \\ [(-1)\mathbf{2}] \\ [(-1)\mathbf{2}+\mathbf{1}]}} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} = \mathbf{R}$$

Basis for $\mathcal{N}(\mathbf{A}^\intercal)$?

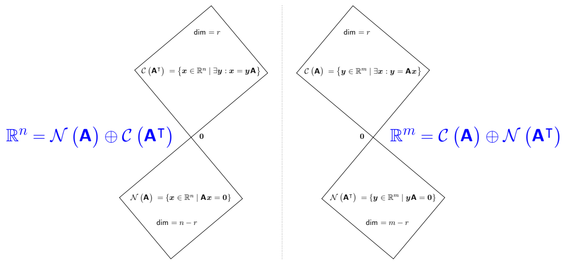
$$[(-1, \quad 0, \quad 1,);]$$

7 Finding a basis of $\mathcal{N}(\mathbf{A}^\intercal)$ by column reduction

$$\begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 1 \\ 0 & 0 & 1 \\ a & b & c \\ d & e & f \end{bmatrix} \xrightarrow{\overset{\boldsymbol{\tau}}{[(1)\mathbf{1}+\mathbf{3}]}} \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \\ a & b & a+c \\ d & e & d+f \end{bmatrix} \xrightarrow{\overset{\boldsymbol{\tau}}{[(1)\mathbf{2}+\mathbf{1}]}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ a+b & b & a+c \\ d+e & e & d+f \end{bmatrix}$$

Basis for $\mathcal{N}(\mathbf{A}^\intercal)$?

8 The Four Fundamental Subspaces



$\underset{m\times n}{\mathbf{A}}$ dimensions?

- $\dim\left(\mathcal{C}(\mathbf{A})\right) = \qquad\qquad = \dim\left(\mathcal{C}(\mathbf{A}^\intercal)\right)$
- $\dim\left(\mathcal{N}(\mathbf{A})\right) =$
- $\dim\left(\mathcal{N}(\mathbf{A}^\intercal)\right) =$

Questions of the Lecture 10

(L-10) QUESTION 1. Find the dimension and construct a basis for the four subspaces associated with each of the matrices

(a) $\mathbf{A} = \begin{bmatrix} 0 & 1 & 4 & 0 \\ 0 & 2 & 8 & 0 \end{bmatrix}$

(b) What is the sum $\dim\mathcal{C}(\mathbf{A}) + \dim\mathcal{N}(\mathbf{A}^\intercal)$? and $\dim\mathcal{C}(\mathbf{A}^\intercal) + \dim\mathcal{N}(\mathbf{A})$?

(c) $\mathbf{U} = \begin{bmatrix} 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

(d) What is the sum $\dim\mathcal{C}(\mathbf{U}) + \dim\mathcal{N}(\mathbf{U}^\intercal)$? and $\dim\mathcal{C}(\mathbf{U}^\intercal) + \dim\mathcal{N}(\mathbf{U})$?

(Strang, 2006, exercise 2 from section 2.4.)

(L-10) QUESTION 2. Describe the four subspaces in three-dimensional space associated with

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

(Strang, 2006, exercise 4 from section 2.4.)

(L-10) QUESTION 3.

(a) If a 7 by 9 matrix has rank 5, what are the dimensions of the four subspaces? What is the sum of all four dimensions?

(b) If a 3 by 4 matrix has rank 3, what are its column space $\mathcal{C}(\mathbf{A})$ and left nullspace $\mathcal{N}(\mathbf{A}^\intercal)$?

(Strang, 2006, exercise 20 from section 2.4.)

(L-10) QUESTION 4. If $\mathbf{A}$ has the same four fundamental subspaces as $\mathbf{B}$, does $\mathbf{A} = c\mathbf{B}$?
(Strang, 2006, exercise 19 from section 2.4.)

(L-10) QUESTION 5. Find the dimension and a basis for the four fundamental subspaces for

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{bmatrix}; \qquad \mathbf{L} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} = \mathbf{AE}; \quad \text{where} \quad \mathbf{E} = \begin{bmatrix} 1 & -2 & 2 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Basado en (Strang, 2006, exercise 3 from section 2.4.)

(L-10) QUESTION 6. Find the dimensions of these vector spaces:

(a) The space of all vectors in $\mathbb{R}^4$ whose components add to zero.
(b) The nullspace of the 4 by 4 identity matrix.
(c) The space of all 4 by 4 matrices

(Strang, 2006, exercise 32 from section 2.3.)

(L-10) QUESTION 7. Without multiplying matrices, find bases for the row and column spaces of $\mathbf{A}$:

$$\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 4 & 5 \\ 2 & 7 \end{bmatrix} \begin{bmatrix} 3 & 0 & 3 \\ 1 & 1 & 2 \end{bmatrix}.$$

How do you know from these shapes that $\mathbf{A}$ is not invertible?
(Strang, 2006, exercise 36 from section 2.4.)

(L-10) QUESTION 8. Which of the following (if any) are subspaces? For any that are not a subspace, give an example of how they violate a property of subspaces.

(a) Given 3×5 matrix $\mathbf{A}$ with full row rank, the set of all solutions to

$$\mathbf{A}\boldsymbol{x} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

(b) All vectors $\boldsymbol{x}$ with $\langle \vec{x} | \vec{y} \rangle = \mathbf{0}$ and $\langle \vec{x} | \vec{z} \rangle = \mathbf{0}$ for some given vectors $\boldsymbol{y}$ and $\boldsymbol{z}$.

(c) All 3×5 matrices with $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$. in their column space.

(d) All 5×3 matrices with $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ in their nullspace.

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(L-10) QUESTION 9. ¿Cuál es el espacio columna $\mathcal{C}(\mathbf{A})$ y el espacio fila $\mathcal{C}(\mathbf{A}^{\intercal})$ de la matriz

$$\mathbf{A} = \begin{bmatrix} 1 & 3 & 5 & -2 \\ 2 & -1 & 3 & -4 \\ -1 & 4 & 2 & 2 \end{bmatrix}$$

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(L-10) QUESTION 10. If $\underset{5\times 4}{\mathbf{A}}$ is a matrix with linearly independent columns, find each of these explicitily:

(a) The nulls space of $\mathbf{A}$.
(b) The dimension of the left null space $\mathcal{N}(\mathbf{A}^{\intercal})$.
(c) One particular solution $\boldsymbol{x}_p$ to the system $\mathbf{A}\boldsymbol{x} = \mathbf{A}_{|2}$.
(d) The general (complete) solution to $\mathbf{A}\boldsymbol{x} = \mathbf{A}_{|2}$.
(e) The reduced echelon form $\mathbf{R}$ of $\mathbf{A}$.

(L-10) QUESTION 11. Verdadero o falso

(a) Si una matriz es cuadrada $(m = n)$, entonces el espacio columna es igual al espacio fila.
(b) La matriz $\mathbf{A}$ y la matriz $(-\mathbf{A})$ comparten los mismos cuatro sub-espacios fundamentales.
(c) Si $\mathbf{A}$ y $\mathbf{B}$ comparten los mismos cuatro sub-espacios fundamentales, entonces $\mathbf{A}$ es un múltiplo de $\mathbf{B}$.
(d) Indique si la siguiente aseveración es verdadera o falsa. Si es verdadera explique el motivo, si es falsa encuentre un contraejemplo: *"Un sistema con n ecuaciones y n incógnitas es resoluble cuando las columnas de la matriz de coeficientes son independientes."*

(L-10) QUESTION 12. Se conoce la siguiente información sobre $\mathbf{A}$:

$$\mathbf{A}\boldsymbol{v} = \mathbf{A}\begin{pmatrix} 1 \\ -2 \\ 3 \\ 1 \end{pmatrix} = \begin{pmatrix} -6 \\ 3 \end{pmatrix}; \qquad \text{y que} \qquad \mathbf{A}\boldsymbol{w} = \mathbf{A}\begin{pmatrix} 3 \\ -1 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} -18 \\ 9 \end{pmatrix}.$$

De hecho, $\mathbf{A}\boldsymbol{x}$ es siempre algún múltiplo del vector $(-2, \quad 1,)$ sea cual sea el vector $\boldsymbol{x} \in \mathbb{R}^4$.

(a) ¿Cuál es el orden y el rango de $\mathbf{A}$?
(b) ¿Cuál es la dimensión del espacio nulo $\mathcal{N}(\mathbf{A})$?
(c) ¿Cuál es la dimensión del espacio fila $\mathcal{C}(\mathbf{A}^{\intercal})$?
(d) ¿Cuál es la dimensión del espacio nulo por la izquierda $\mathcal{N}(\mathbf{A}^{\intercal})$?
(e) Encuentre una solución $\boldsymbol{x}$ no nula al sistema $\mathbf{A}\boldsymbol{x} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

(L-10) QUESTION 13. Consider the matrix $\mathbf{A}$ with its column reduced echelon form $\mathbf{R}$ computed by gaussian elimination without permutations:

$$\left[\frac{\mathbf{A}}{\mathbf{I}}\right] = \left[\begin{array}{rrrrr} 1 & -1 & 5 & 1 & 0 \\ 2 & 1 & 4 & 2 & 1 \\ 3 & 0 & 9 & 3 & 1 \\ -1 & -1 & -1 & -1 & 2 \\ \hline 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{rrrrr} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ \hline 3/8 & 2/8 & -3 & -1 & -1/8 \\ -5/8 & 2/8 & 2 & 0 & -1/8 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ -1/8 & 2/8 & 0 & 0 & 3/8 \end{array}\right] = \left[\frac{\mathbf{R}}{\mathbf{E}}\right]$$

(a) What is the rank of $\mathbf{A}$? What are the dimensions of the column space $\mathcal{C}(\mathbf{A})$, the row space $\mathcal{C}(\mathbf{A}^{\intercal})$ and the nullspace $\mathcal{N}(\mathbf{A})$?
(b) Find a basis of the row space $\mathcal{C}(\mathbf{A}^{\intercal})$.
(c) Find a basis for the column space $\mathcal{C}(\mathbf{A})$.
(d) Find a basis for the nullspace $\mathcal{N}(\mathbf{A})$.
(e) Write down $\mathbf{A}_{|3}$ as a linear combination of $\mathbf{A}_{|1}$, $\mathbf{A}_{|2}$, $\mathbf{A}_{|4}$ and $\mathbf{A}_{|5}$.

(L-10) QUESTION 14. Construct a matrix whose nullspace consists of all combinations of $(2, 2, 1, 0,)$ and $(3, 1, 0, 1,)$.
(Strang, 2006, exercise 60 from section 2.2.)

(L-10) QUESTION 15. Construct a matrix whose nullspace consists of all multiples of $(4, 3, 2, 1)^\intercal$
(Strang, 2006, exercise 61 from section 2.2.)

(L-10) QUESTION 16.

(a) Suponga que el producto de **A** y **B** es la matriz nula: $\mathbf{AB} = \mathbf{0}$. Entonces el espacio (I)_______________ de la matriz **A** contiene el espacio (II)_______________ de la matriz **B**. También el espacio (III)_______________ de la matriz **B** contiene el espacio (IV)_______________ de la matriz **A**. (incluya los nombres de los cuatro espacios fundamentales en los lugares apropiados)

   (I)_______________, (II)_______________, (III)_______________, (IV)_______________

(b) Suponga que la matriz **A** es de dimensiones 5 por 7 con rango r, y **B** es de dimensiones 7 por 9 de rango s. ¿Cuáles son las dimensiones de los espacios (I) y (II)? Del hecho de que el espacio (I) contiene el espacio (II), ¿qué sabe acerca de $r + s$?

(L-10) QUESTION 17. By performing column eliminations (and possibly permutations) on the following 4×8 matrix **A**

$$\left[\frac{\mathbf{A}}{\mathbf{I}}\right] = \left[\begin{array}{rrrrrrrr}
1 & 2 & -2 & 1 & -5 & 0 & 2 & -3 \\
-1 & -2 & 1 & -2 & 3 & 0 & -2 & 0 \\
1 & 2 & -2 & 1 & -5 & 1 & 1 & 1 \\
0 & 0 & 2 & 2 & 4 & 0 & 0 & 2 \\
\hline
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{array}\right]$$

$$\rightarrow \left[\begin{array}{rrrrrrrr}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
\hline
2/4 & -2 & -2/4 & -3 & 1 & 0 & -2 & 3/4 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
2/4 & 0 & 2/4 & -1 & -2 & 0 & 0 & 3/4 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
1 & 0 & 2 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
-2/4 & 0 & -2/4 & 0 & 0 & 0 & 0 & -1/4
\end{array}\right] = \left[\frac{\mathbf{R}}{\mathbf{E}}\right]$$

(a) What is the rank of **A**?
(b) What are the dimensions of the 4 fundamental subspaces?

(c) How many solutions does $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$ have? Does it depend on $\boldsymbol{b}$? Justify.
(d) Are the rows of **A** linearly independent? Why?
(e) Give a basis of $\mathcal{N}(\mathbf{A})$.
(f) Give a basis of $\mathcal{N}(\mathbf{A}^\intercal)$.
(g) Give, if possible, matrix $\left[\mathbf{A}_{|1};\ \mathbf{A}_{|3};\ \mathbf{A}_{|6};\ \mathbf{A}_{|7};\right]^{-1}$
(h) Give, if possible, matrix $\left[\mathbf{A}_{|1};\ \mathbf{A}_{|3};\ \mathbf{A}_{|6};\ \mathbf{A}_{|8};\right]^{-1}$

Based on *MIT Course 18.06 Quiz 1, October 4, 2004*

(L-10) QUESTION 18. Consider the 5 by 3 matrix **R** (in its column reduced echelon form) with three pivots ($r = 3$).

(a) What is its null space $\mathcal{N}(\mathbf{R})$?
(b) Consider the 10 by 3 block matrix; $\mathbf{B} = \begin{bmatrix} \mathbf{R} \\ 2\mathbf{R} \end{bmatrix}$. What is its column reduced echelon form? What is its rank?
(c) Consider the 10 by 6 block matrix; $\mathbf{C} = \begin{bmatrix} \mathbf{R} & \mathbf{R} \\ \mathbf{R} & \mathbf{0} \end{bmatrix}$. What is its column reduced echelon form?
(d) What is the rank of **C**?
(e) What is the dimension of the null space of $\mathbf{C}^\intercal$; $\dim \mathcal{N}(\mathbf{C}^\intercal)$?

(L-10) QUESTION 19. Consider the linear system $\mathbf{A}\boldsymbol{x} = \begin{pmatrix} 2 \\ 4 \\ 2 \end{pmatrix}$

with solution $\left\{ \boldsymbol{x} \in \mathbb{R}^3 \;\middle|\; \exists\, c, d \in \mathbb{R} \text{ such that } \boldsymbol{x} = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} + c \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + d \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$.

(a) (1^{pts}) Find the dimension of the row space of **A**. Explain your answer.
(b) (1^{pts}) Construct the matrix **A**. Explain your answer.
(c) (0.5^{pts}) For which right hand side vectors $\boldsymbol{b}$ the system $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$ is solvable?

(L-10) QUESTION 20. True or false (give a good reason)?

(a) If the columns of a matrix are dependent, so are the rows.
(b) The column space of a 2 by 2 matrix is the same as its row space.
(c) The column space of a 2 by 2 matrix has the same dimension as its row space.
(d) The columns of a matrix are a basis for the column space.

(Strang, 2006, exercise 28 from section 2.3.)

(L-10) QUESTION 21. Let **A** be any matrix and **R** its **row reduced** echelon form. Answer True or False to the statements below and briefly explain. (Note, if there are any counterexamples to a statement below you must choose false for that statement.)

(a) If $\boldsymbol{x}$ is a solution to $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$ then $\boldsymbol{x}$ must be a solution to $\mathbf{R}\boldsymbol{x} = \boldsymbol{b}$.
(b) If $\boldsymbol{x}$ is a solution to $\mathbf{A}\boldsymbol{x} = \mathbf{0}$ then $\boldsymbol{x}$ must be a solution to $\mathbf{R}\boldsymbol{x} = \mathbf{0}$.
(c) What would be your anwsers if **R** is the **column** reduced form of **A**?

1 Highlights of **Lesson**

Highlights of Lesson

- Bases of new vector spaces
- Rank one matrices
- Free variables

2 A new vector space

$\mathbb{R}^{3\times 3}$: All matrices of order 3×3! $\quad \mathbf{A} + \mathbf{B}$; $\quad c\mathbf{A}$; $\quad \underset{3\times 3}{\mathbf{0}}$

subspaces of $\mathbb{R}^{3\times 3}$

- $\mathcal{U}$: Upper triangular matrices
- $\mathcal{S}$: Symmetric matrices
- $\mathcal{U} \cap \mathcal{S}$: The intersection (the vectors that are in both $\mathcal{S}$ and $\mathcal{U}$):

What are the dimensions of these subspaces?

Is $\mathcal{U} \cup \mathcal{S}$ a subspace?

Let $\mathcal{U} + \mathcal{S}$ be the set of all sums of vectors in $\mathcal{U}$ plus vectors in $\mathcal{S}$; then $\mathcal{U} + \mathcal{S} =$?

3 Rank one matrices

$$\mathbf{A} = \begin{bmatrix} 1 & 4 & 5 \\ & & \end{bmatrix}$$

- tell me a basis for the row space:
- tell me a basis for the column space

What's the dimension of $\mathcal{C}(\mathbf{A})$, $\mathcal{C}(\mathbf{A}^\intercal)$ and $\mathrm{rg}(\mathbf{A})$?

$$\mathbf{A} = \begin{bmatrix} 1 & 4 & 5 \\ & & \end{bmatrix} = \begin{bmatrix} 1 \\ \end{bmatrix} \begin{bmatrix} 1 & 4 & 5 \end{bmatrix}$$

Every rank one matrix has the form: a column times a row.

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}_{|1} \end{bmatrix} \begin{bmatrix} {}_{1|}\mathbf{A} \end{bmatrix}^\intercal = \text{column matrix times row matrix}$$

4 Rank one matrices

Think about the following subset of $\mathbb{R}^4$:

$$\mathcal{S} = \left\{ \boldsymbol{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix} \in \mathbb{R}^4 \;\middle|\; v_1 + v_2 + v_3 + v_4 = 0 \right\}$$

Is $\mathcal{S}$ a subspace?

What's the dimension? can you tell me a basis?

$\mathcal{S}$ is the null space of a matrix $\mathbf{A}$ $(\mathbf{A}\boldsymbol{v} = 0)$. . . Which matrix?

5 Rank one matrices

$$\underset{1\times 4}{\mathbf{A}} = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}; \qquad \operatorname{rg}(\mathbf{A}) = \qquad\qquad \mathcal{S} = \mathcal{N}(\mathbf{A})$$

- $\dim \mathcal{S} = \dim \mathcal{N}(\mathbf{A}) =$
- basis of $\mathcal{S} = \mathcal{N}(\mathbf{A})$?

$$\left[\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}; \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}; \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}; \right]$$

- $\dim \mathcal{C}(\mathbf{A}^\intercal) =$
- $\dim \mathcal{N}(\mathbf{A}^\intercal) =$ basis $\mathcal{N}(\mathbf{A}^\intercal)$?

6 A problem from Microeconomics

Solve Y in terms of X to get PPF

$$\begin{cases} X & = 4L_x \\ \quad Y & = 3L_y \\ \qquad L_x + L_y & = 80 \end{cases} \rightarrow \begin{cases} X \quad - 4L_x & = 0 \\ \quad Y \qquad - 3L_y & = 0 \\ \qquad L_x + \;\; L_y & = 80 \end{cases}$$

("in terms of" X means X free)

$$\left[\begin{array}{cccc|c} 1 & 0 & -4 & 0 & 0 \\ 0 & 1 & 0 & -3 & 0 \\ 0 & 0 & 1 & 1 & -80 \\ \hline 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow[{[(3)\mathbf{2}+\mathbf{4}]}]{\substack{\boldsymbol{\tau} \\ [(4)\mathbf{1}+\mathbf{3}]}} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & -80 \\ \hline 1 & 0 & 4 & 0 & 0 \\ 0 & 1 & 0 & 3 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow[{[(80)\mathbf{3}+\mathbf{5}]}]{\substack{\boldsymbol{\tau} \\ [(-1)\mathbf{3}+\mathbf{4}]}} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ \hline 1 & 0 & 4 & -4 & 320 \\ 0 & 1 & 0 & 3 & 0 \\ 0 & 0 & 1 & -1 & 80 \\ 0 & 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\begin{pmatrix} X \\ Y \\ L_x \\ L_y \end{pmatrix} = \begin{pmatrix} 320 \\ 0 \\ 80 \\ 0 \end{pmatrix} + a \begin{pmatrix} -4 \\ 3 \\ -1 \\ 1 \end{pmatrix} \Rightarrow a = L_y \Rightarrow \begin{pmatrix} X \\ Y \\ L_x \\ L_y \end{pmatrix} = \begin{pmatrix} 320 - 4L_y \\ 3L_y \\ 80 - L_y \\ L_y \end{pmatrix} \quad \text{"in terms of" } L_y$$

7 Free variable

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ \hline 1 & 0 & 4 & -4 & 320 \\ 0 & 1 & 0 & 3 & 0 \\ 0 & 0 & 1 & -1 & 80 \\ 0 & 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow[{[(-320)\mathbf{4}+\mathbf{5}]}]{\substack{\boldsymbol{\tau} \\ [(\frac{-1}{4})\mathbf{4}]}} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ \hline 1 & 0 & 4 & 1 & 0 \\ 0 & 1 & 0 & -3/4 & 240 \\ 0 & 0 & 1 & 1/4 & 0 \\ 0 & 0 & 0 & -1/4 & 80 \\ \hline 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\begin{pmatrix} X \\ Y \\ L_x \\ L_y \end{pmatrix} = \begin{pmatrix} 0 \\ 240 \\ 0 \\ 80 \end{pmatrix} + a \begin{pmatrix} 1 \\ -\frac{3}{4} \\ \frac{1}{4} \\ -\frac{1}{4} \end{pmatrix} \Rightarrow a = X \Rightarrow \begin{pmatrix} X \\ Y \\ L_x \\ L_y \end{pmatrix} = \begin{pmatrix} X \\ 240 - \frac{3}{4}X \\ \frac{1}{4}X \\ 80 - \frac{1}{4}X \end{pmatrix}$$

"in terms of" X

8 Free variables

$$\begin{cases} x + 2y - z + w = -1 \\ -x - 2y + 3z + 5w = -5 \\ -x - 2y - z - 7w = 7 \end{cases}$$

1. Solve in terms of y and w
2. Solve in terms of x and w
3. Solve in terms of x and z
4. Solve in terms of x and y

$$\left[\begin{array}{cccc|c} 1 & 2 & -1 & 1 & -1 \\ -1 & -2 & 3 & 5 & -5 \\ -1 & -2 & -1 & -7 & 7 \\ \hline 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{\substack{\boldsymbol{\tau} \\ [(-2)\mathbf{1}+\mathbf{2}] \\ [(1)\mathbf{1}+\mathbf{3}] \\ [(-1)\mathbf{1}+\mathbf{4}] \\ [(1)\mathbf{1}+\mathbf{5}]}} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 2 & 6 & -6 \\ -1 & 0 & -2 & -6 & 6 \\ \hline 1 & -2 & 1 & -1 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{\substack{\boldsymbol{\tau} \\ [(-3)\mathbf{3}+\mathbf{4}] \\ [(3)\mathbf{3}+\mathbf{5}]}} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 2 & 0 & 0 \\ -1 & 0 & -2 & 0 & 0 \\ \hline 1 & -2 & 1 & -4 & 4 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -3 & 3 \\ 0 & 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{cc|c} -2 & -4 & 4 \\ 1 & 0 & 0 \\ 0 & -3 & 3 \\ 0 & 1 & 0 \end{array}\right] \begin{cases} \xrightarrow{\substack{\boldsymbol{\tau} \\ [(\frac{-1}{2})\mathbf{1}] \\ [(4)\mathbf{1}+\mathbf{2}] \\ [(-4)\mathbf{1}+\mathbf{3}]}} \left[\begin{array}{cc|c} 1 & 0 & 0 \\ \frac{-1}{2} & -2 & 2 \\ 0 & -3 & 3 \\ 0 & 1 & 0 \end{array}\right] \xrightarrow{\substack{\boldsymbol{\tau} \\ [(1)\mathbf{2}+\mathbf{3}] \\ [(\frac{-1}{3})\mathbf{2}]}} \left[\begin{array}{cc|c} 1 & 0 & 0 \\ \frac{-1}{2} & \frac{2}{3} & 0 \\ 0 & 1 & 0 \\ 0 & \frac{-1}{3} & 1 \end{array}\right] \\ \xrightarrow{\substack{\boldsymbol{\tau} \\ [(\frac{-1}{2})\mathbf{1}] \\ [(4)\mathbf{1}+\mathbf{2}] \\ [(-4)\mathbf{1}+\mathbf{3}]}} \left[\begin{array}{cc|c} 1 & 0 & 0 \\ \frac{-1}{2} & -2 & 2 \\ 0 & -3 & 3 \\ 0 & 1 & 0 \end{array}\right] \xrightarrow{\substack{\boldsymbol{\tau} \\ [(\frac{-1}{2})\mathbf{2}] \\ [(\frac{1}{2})\mathbf{2}+\mathbf{1}] \\ [(-2)\mathbf{2}+\mathbf{3}]}} \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{3}{4} & \frac{3}{2} & 0 \\ \frac{-1}{4} & \frac{-1}{2} & 1 \end{array}\right] \end{cases}$$

Questions of the Optional Lecture 1

(L-Opt-1) Question 1.

(a) What is the smallest subspace of 3 by 3 matrices which contains all symmetric matrices and all lower triangular matrices?
(b) What is the largest subspace which is contained in both of those subspaces?

(Strang, 1988, exercise 4 from section 1.2.)

(L-Opt-1) Question 2. For each of these statements, say whether the claim is true or false and give a brief justification.

(a) **True/False:** The set of 3×3 non-invertible matrices forms a subspace of the set of all 3×3 matrices.
(b) **True/False:** If the system $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$ has no solution then **A** does not have full row rank.
(c) **True/False:** There exist $n \times n$ matrices **A** and **B** such that **B** is not invertible but **AB** is invertible.
(d) **True/False:** For any permutation matrix **P** , we have that $\mathbf{P}^2 = \mathbf{I}$.

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(L-Opt-1) Question 3.

(a) Sean los vectores $\boldsymbol{u}$, $\boldsymbol{v}$ y $\boldsymbol{w}$ en $\mathbb{R}^7$ ¿Cuál es la dimensión (o cuáles son las posibles dimensiones) del espacio generado por estos tres vectores?
(b) Sea una matriz cuadrada **A**. Si su espacio nulo $\mathcal{N}(\mathbf{A})$ está compuesto únicamente por el vector nulo $\mathbf{0}$, ¿Cuál es el espacio nulo de su traspuesta (espacio nulo por la izquierda $\mathcal{N}(\mathbf{A}^\intercal)$?
(c) Piense en el espacio vectorial de todas las matrices de orden 5 por 5, $\mathbb{R}^{5\times5}$. Piense en el subconjunto de matrices 5 por 5 que son invertibles ¿es este subconjunto un sub-espacio vectorial? Si lo es, explique el motivo; si no lo es encuentre un contraejemplo.
(d) Indique si la siguiente aseveración es verdadera o falsa. Si es verdadera explique el motivo, si es falsa encuentre un contraejemplo: *"Si* $\mathbf{B}^2=\mathbf{0}$*, entonces necesariamente* $\mathbf{B}=\mathbf{0}$*"*
(e) Si intercambio dos columnas de la matriz **A** ¿qué espacios fundamentales siguen siendo iguales?
(f) Si intercambio dos filas de la matriz **A** ¿qué espacios fundamentales siguen siendo iguales?
(g) ¿Por qué el vector $\boldsymbol{v}=\begin{pmatrix}1\\2\\3\end{pmatrix}$ no puede estar en el espacio nulo de una matriz **A** y simultáneamente ser una fila de dicha matriz?

(L-Opt-1) Question 4. Empleando la definición de sub-espacio vectorial, verifique si los siguientes subconjuntos son sub-espacios vectoriales del espacio vectorial que los contiene.

(a) $\mathcal{V}$ es el espacio vectorial de todas las matrices 2×2 de números reales, con las operaciones habituales de suma y producto por un escalar; y el conjunto $\mathcal{W}$ son todas las matrices de la forma
$$\begin{bmatrix}a & b\\0 & b\end{bmatrix}$$
donde a y b son números reales.
(b) $\mathcal{V}$ es el espacio vectorial $C[0,1]$ de todas las funciones continuas en el intervalo $[0,1]$; y el conjunto $\mathcal{W}$ son todas las funciones $f\in C[0,1]$ tales que $f(0)=2$.

(L-Opt-1) Question 5. Encuentre una base (de dimensión infinita) para el espacio de todos los polinomios

$$\mathcal{P}=\Big\{a_nx^n+a_{n-1}x^{n-1}+\cdots+a_1x+a_0 \;\big|\; \text{para todo } n\Big\}.$$

(L-Opt-1) Question 6. ¿Cuál es la dimensión de los siguientes espacios?

(a) El conjunto de matrices simétricas de orden 2×2, $\mathbf{A}=\mathbf{A}^\intercal$.
$$\mathbf{A}=\left[\begin{array}{cc}a & b\\ b & d\end{array}\right],$$
(b) El conjunto de matrices simétricas de orden 2×2
$$\mathbf{A}=\left[\begin{array}{cc}a & b\\ c & d\end{array}\right],$$
tales que $a+d=0$.
(c) El conjunto de vectores de $\mathbb{R}^4$ de la forma
$\Big\{\Big(x,\ y,\ (x-3y),\ (2y-x),\Big)\ \mid\ x,y\in\mathbb{R}\Big\}.$

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