Mathematics II

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You can find the last version of these course materials at

https://github.com/mbujosab/MatematicasII/tree/main/Eng

Contents

Part III

Subspaces and systems of linear equations

LECTURE 6: Introduction to vector spaces and sub-spaces

Lecture 6

(**Lecture** 6) S-1 Highlights of Lesson 6

Highlights of *Lesson 6*

- Introduction to vector spaces and sub-spaces

F1

(**Lecture** 6) S-2 Introduction

What are the main operations that we do with vectors?

- We add them: $\boldsymbol{v}+\boldsymbol{w}$
- We multiply them by numbers, usually called scalars: $\lambda \boldsymbol{v}$

F2

(**Lecture** 6) S-3 Vector space: definition

A *vector space* is a set $\mathcal{V}$ together with *two operations*

Addition $(\vec{x}+\vec{y})$**:** $\mathcal{V}\times\mathcal{V}\to\mathcal{V}$
It associates with each pair $\vec{x}$, $\vec{y}$ another element of $\mathcal{V}$ called $\vec{x}+\vec{y}$

Scalar product $(\alpha\vec{x})$**:** $\mathbb{R}\times\mathcal{V}\to\mathcal{V}$
It associates with each pair α, $\vec{x}$ another element of $\mathcal{V}$ called $\alpha\vec{x}$

satisfying:

- $\vec{x}+\vec{y}=\vec{y}+\vec{x}$
- $\vec{x}+(\vec{y}+\vec{z})=(\vec{x}+\vec{y})+\vec{z}$
- There exists a unique $\vec{0}$ such that $\vec{x}+\vec{0}=\vec{x}$
- For each $\vec{x}$ there is a unique $-\vec{x}$ such that $\vec{x}+(-\vec{x})=\vec{0}$

- $\alpha(\vec{x}+\vec{y})=\alpha\vec{x}+\alpha\vec{y}$
- $(\alpha+\beta)\vec{x}=\alpha\vec{x}+\beta\vec{x}$
- $(\alpha\cdot\beta)\vec{x}=\alpha(\beta\vec{x})$
- $1\vec{x}=\vec{x}$

F3

(**Lecture** 6) S-**4** Vector space

- A vector space is a set of mathematical *objects*
  (they could be numbers, lists of numbers, matrices, functions, etc...)
- and two operations:
  - *vector addition*
  - *scalar multiplication*.

  satisfying the eigth above axioms.
- The elements of a vector space are called vectors.

For us, scalars will be always real numbers ($\mathbb{R}$).

F4

(**Lecture** 6) S-**5** Examples: $\mathbb{R}^2$

$\mathbb{R}^2$: The space of all vectors with two components

$$\begin{pmatrix} 3 \\ 2 \end{pmatrix} ; \quad \begin{pmatrix} 0 \\ 0 \end{pmatrix} ; \quad \begin{pmatrix} \pi \\ e \end{pmatrix} ; \quad \begin{pmatrix} \text{1st comp.} \\ \text{2nd comp} \end{pmatrix}$$

$\mathbb{R}^2$ is represented by the usual xy plane

F5

(**Lecture** 6) S-**6** More examples

$\mathbb{R}^3$: the space af all vectors with *three* components

$$\begin{pmatrix} 3 \\ 2 \\ 0 \end{pmatrix}$$

$\mathbb{R}^1$: lists with only one real number: $(0,)$ $(\pi,)$ $(a,)$ $(7,)$
$\mathbb{R}^n$: the space af all vectors with n components

F6

(**Lecture** 6) S-**7** Subspaces

A subspace $\mathcal{W}$ of the vector space $\mathcal{V}$
is a non-empty subset of $\mathcal{V}$ (with the same operations of $\mathcal{V}$) such

that for any $\vec{v}$ and $\vec{w}$ in $\mathcal{W}$ and scalars c and d:

- $(\vec{v} + \vec{w})$ is in $\mathcal{W}$
- $(c \cdot \vec{v})$ is in $\mathcal{W}$

$$\boxed{\text{All linear combinations } (c \cdot \vec{v} + d \cdot \vec{w}) \text{ stay in } \mathcal{W}}$$

$\mathcal{W} \subset \mathcal{V}$ is a subspace if it is closed under both operations.
A subspace of $\mathcal{V}$ is a vector space inside the vector space $\mathcal{V}$.

F7

(**Lecture** 6) **S-8** **Examples**

Which ones of the following subsets of $\mathbb{R}^2$ are subspaces?

- The first quarter-plane
- Any line in $\mathbb{R}^2$ through $(0, 0)$
- A line in $\mathbb{R}^2$ that doesn't contain the origin
- $\{\mathbf{0}\}$: the set that consists only of a zero vector $\mathbf{0}$

Every subspace *has its own zero vector* $\mathbf{0}$

F8

(**Lecture** 6) **S-9** **List of all possible subspaces of $\mathbb{R}^2$**

1. The whole space (plane) $\mathbb{R}^2$
2.
3.

and the subspaces of $\mathbb{R}^3$? gráfico 3D

F9

(**Lecture** 6) **S-10** **Union and intersection of subspaces**

Let $\mathcal{S}$ and $\mathcal{T}$ be two subspaces

- $\mathcal{S} \cup \mathcal{T}$: their union (take the vector in $\mathcal{S}$ and $\mathcal{T}$ all together)

  Is the union a subspace?

- $\mathcal{S} \cap \mathcal{T}$: the intersection (vectors belonging to both $\mathcal{S}$ and $\mathcal{T}$)

  Is the intersection a subspace? (proof?)

F10

The lecture ends here

Questions of the Lecture 6

(L-6) Question 1.

(a) Find a subset W in $\mathbb{R}^2$ ($W \subseteq \mathbb{R}^2$) closed under vector addition (if $\boldsymbol{v}, \boldsymbol{w} \in W$, then $\boldsymbol{v} + \boldsymbol{w} \in W$), but not under scalar multiplication ($c\boldsymbol{v}$ is not necessarily in W).

(b) Find a subset W in $\mathbb{R}^2$ ($W \subseteq \mathbb{R}^2$) closed under scalar multiplication (if $\boldsymbol{v}, \boldsymbol{w} \in W$, then $c\boldsymbol{v} \in W$) but not under vector addition ($\boldsymbol{v} + \boldsymbol{w} \in W$ is not necessarily in W).

(Strang, 2006, exercise 1 from section 2.1.)

(L-6) Question 2. Consider $\mathbb{R}^2$ as a vector space. Which of the following are subspaces and which are not? If not, why not?

(a) $\{(a,\ a^2,) \mid a \in \mathbb{R}\}$
(b) $\{(b,\ 0,) \mid b \in \mathbb{R}\}$
(c) $\{(0,\ c,) \mid c \in \mathbb{R}\}$
(d) $\{(m,\ n,) \mid m, n \in \mathbb{Z}\}$ where $\mathbb{Z}$ is the set of integer numbers.
(e) $\{(d,\ e,) \mid d, e \in \mathbb{R},\ d \cdot e = 0\}$

(f) $\{(f,\ f,) \mid f \in \mathbb{R}\}$

(L-6) QUESTION 3. Why isn't $\mathbb{R}^2$ a subspace of $\mathbb{R}^3$?
(Strang, 2006, exercise 31 from section 2.1.)

(L-6) QUESTION 4. Let P be the plane in $\mathbb{R}^3$ defined by the equation

$$x - y - z = 3.$$

Find two vectors in P and show that their sum is not in P.

(L-6) QUESTION 5. Show that for any $\boldsymbol{b} \neq \mathbf{0}$, the solution set $\{\boldsymbol{x} \mid \mathbf{A}\boldsymbol{x} = \boldsymbol{b}\}$ does not form a subspace.

(L-6) QUESTION 6. Consider the set $\mathbb{R}^{2\times 2}$ as a vector space. Let

$$\mathbf{A} = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}; \qquad \mathbf{B} = \begin{bmatrix} 0 & 0 \\ 0 & -3 \end{bmatrix};$$

(a) Name a subspace containing $\mathbf{A}$ but not $\mathbf{B}$.
(b) Name a subspace containing $\mathbf{B}$ but not $\mathbf{A}$.
(c) Is there a subspace containing $\mathbf{A}$ and $\mathbf{B}$ but not the 2×2 identity matrix $\underset{n\times n}{\mathbf{I}}$?

(L-6) QUESTION 7. Consider the set $\mathbb{R}^{n\times n}$ as a vector space. Which of the following are subspaces?
(a) The symmetric matrices, $\mathcal{S} = \{\mathbf{A} \in \mathbb{R}^{n\times n} \mid {}_{j|}\mathbf{A} = \mathbf{A}_{|j}\}$
(b) The non-symmetric matrices, $\mathcal{NS} = \{\mathbf{A} \in \mathbb{R}^{n\times n} \mid \mathbf{A}^{\mathsf{T}} \neq \mathbf{A}\}$
(c) The skew-symmetric matrices, $\mathcal{AS} = \{\mathbf{A} \in \mathbb{R}^{n\times n} \mid \mathbf{A}^{\mathsf{T}} = -\mathbf{A}\}$

(L-6) QUESTION 8.
(a) The intersection of two planes through $(0,0,0,)$ is probably a____________, but it could be a ____________.
(b) The intersection of a plane through $(0,0,0,)$ with a line through $(0,0,0,)$ is probably a ____________, but it could be a ____________.
(c) If $\mathcal{S}$ and $\mathcal{T}$ are subspaces of $\mathbb{R}^5$, their intersection $\mathcal{S} \cap \mathcal{T}$ (vectors in both sub-spaces) is a subspace of $\mathbb{R}^5$. Check the requirements on $\boldsymbol{x} + \boldsymbol{y}$ and $c\boldsymbol{x}$.
(Strang, 2006, exercise 18 from section 2.1.)

(L-6) QUESTION 9. Which of the following subsets of $\mathbb{R}^3$ are actually subspaces?
(a) The plane of vectors $\boldsymbol{b} = (b_1,\ b_2,\ b_3,)$ with first component $b_1 = 0$.
(b) The plane of vectors $\boldsymbol{b} = (b_1,\ b_2,\ b_3,)$ with first component $b_1 = 1$.
(c) The vectors $\boldsymbol{b}$ with $b_2 b_3 = 0$ (this is the union of two subspaces. the plane $b_2 = 0$ and the plane $b_3 = 0$).
(d) The solitary vector $\boldsymbol{b} = \mathbf{0}$.
(e) All combinations of two given vectors $(1,\ 1,\ 0,)$ and $(2,\ 0,\ 1,)$.
(f) The vectors $(b_1,\ b_2,\ b_3,)$ that satisfies $b_3 - b_2 + 3b_1 = 0$.
(Strang, 2006, exercise 2 from section 2.1.)

One more. . . not so easy

(L-6) QUESTION 10. Addition and scalar multiplication are required to satisfy these eight rules:

1. $\boldsymbol{x} + \boldsymbol{y} = \boldsymbol{y} + \boldsymbol{x}$.
2. $\boldsymbol{x} + (\boldsymbol{y} + \boldsymbol{z}) = (\boldsymbol{x} + \boldsymbol{y}) + \boldsymbol{z}$.
3. There is a unique $\mathbf{0}$ ("zero vector") such that $\boldsymbol{x} + \mathbf{0} = \boldsymbol{x}$ for all $\boldsymbol{x}$.
4. For each $\boldsymbol{x}$ there is a unique vector $-\boldsymbol{x}$ such that $\boldsymbol{x} + (-\boldsymbol{x}) = \mathbf{0}$.
5. $1\boldsymbol{x} = \boldsymbol{x}$.
6. $(a \cdot b)\boldsymbol{x} = a(b\boldsymbol{x})$.

7. $a(\boldsymbol{x}+\boldsymbol{y}) = a\boldsymbol{x} + a\boldsymbol{y}$.

8. $(a+b)\boldsymbol{x} = a\boldsymbol{x} + b\boldsymbol{x}$.

(a) Suppose addition in $\mathbb{R}^2$ adds an extra 1 to each component, so that $(3,1,)+(5,0,) = (9,2,)$ instead of $(8,1,)$. With scalar multiplication unchanged, which rules are broken?

(b) Show that the set of all positive real numbers is a vector space, when the addition and multiplication are redefined to be as follows:

- $\boldsymbol{x}+\boldsymbol{y} = xy$
- $\boldsymbol{x} = x^c$

What is the "zero vector" $\mathbf{0}$?:

(c) Suppose $(x_1,\ x_2,) + (y_1,\ y_2,)$ is defined to be $\Big((x_1+y_2),\ (x_2+y_1),\Big)$; With the usual $c\boldsymbol{x} = (cx_1,\ cx_2,)$. which of the eight conditions are not satisfied?

(Strang, 1988, exercise 5 from section 2.1.)

End of Questions of the Lecture 6

LECTURE 7: Solving $\mathbf{A}\boldsymbol{x} = 0$

Lecture 7

(**Lecture** 7) S-1 Highlights of Lesson 7

Highlights of *Lesson 7*

- Null space of a matrix $\mathbf{A}$: solving $\mathbf{A}\boldsymbol{x} = \mathbf{0}$

A natural algorithm for solving $\mathbf{A}\boldsymbol{x} = 0$?
by elimination (**column** reduction)

- **Column** (pre)echelon form
- pivot (*or endogenous*) variables and free (*or exogenous*) variables
- Special solutions

F11

(**Lecture** 7) S-2 The subspaces of a matrix: the null space $\mathcal{N}(\mathbf{A})$

$$\mathbf{A}\boldsymbol{x} = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \\ 4 & 1 & 5 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\boxed{\mathcal{N}(\mathbf{A}) \text{ is the set of vectors } \boldsymbol{x} \text{ that solve } \mathbf{A}\boldsymbol{x} = \mathbf{0}.}$$

$\mathcal{N}(\mathbf{A})$ is *subset* of ¿$\mathbb{R}^?$?
Find some solutions. Find all solutions

And what does it look like? (graph)

F12

(**Lecture** 7) S-3 Is the null space $\mathcal{N}(\mathbf{A})$ a subspace?

We should check that the set of all solutions to $\mathbf{A}\boldsymbol{v} = \mathbf{0}$ is a subspace.
We should check that for any $a, b \in \mathbb{R}$
If $\mathbf{A}\boldsymbol{v} = \mathbf{0}$ and $\mathbf{A}\boldsymbol{w} = \mathbf{0}$, then

Therefore $\mathcal{N}(\mathbf{A})$ is a subspace

F13

(**Lecture** 7) S-4 Solutions of a general linear system

Let me change the right-hand side to (one, two, three, four).

$$\mathbf{A}\boldsymbol{x} = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \\ 4 & 1 & 5 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$$

That is a very special right-hand side. And we know that there are some solutions

Do they form a subspace?
Is the zero vector $\mathbf{0}$ a solution?

F14

(**Lecture** 7) S-5 A natural algorithm for finding $\mathcal{N}(\mathbf{A})$?

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 2 & 2 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 8 & 10 \end{bmatrix}$$

- Which columns are a linear combination of the other columns?
- Elimination will tell us. . .

F15

(**Lecture** 7) S-6 Which columns are a linear combination of the other columns?

$$\left[\frac{\mathbf{A}}{\mathbf{I}}\right] = \left[\begin{array}{cccc} 1 & 2 & 2 & 2 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 8 & 10 \\ \hline 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{array}\right] \longrightarrow \qquad\qquad = \left[\frac{\mathbf{K}}{\mathbf{E}}\right]$$

$$\text{Then} \quad \mathbf{A} \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \mathbf{0} \qquad \Longrightarrow$$

$$\text{and} \quad \mathbf{A} \begin{pmatrix} 2 \\ 0 \\ -2 \\ 1 \end{pmatrix} = \mathbf{0} \qquad \Longrightarrow$$

F16

(**Lecture** 7) S-7 How to compute $\mathcal{N}(\mathbf{A})$: elimination and "special solutions"

$$\boxed{\mathbf{A}\left(\mathbf{I}_{\tau_1\cdots\tau_k}\right)_{|j} = \left(\mathbf{A}_{\tau_1\cdots\tau_k}\right)_{|j}}$$

$$\left[\begin{array}{c}\mathbf{A}\\\hline \mathbf{I}\end{array}\right] = \left[\begin{array}{cccc} 1 & 2 & 2 & 2\\ 2 & 4 & 6 & 8\\ 3 & 6 & 8 & 10\\\hline 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1\end{array}\right] \xrightarrow[{[(-2)\mathbf{1}+\mathbf{4}]}]{\substack{\tau\\ [(-2)\mathbf{1}+\mathbf{2}]\\ [(-2)\mathbf{1}+\mathbf{3}]}} \left[\begin{array}{cccc} 1 & 0 & 0 & 0\\ 2 & 0 & 2 & 4\\ 3 & 0 & 2 & 4\\\hline 1 & -2 & -2 & -2\\ & 1 & & \\ & & 1 & \\ & & & 1\end{array}\right] \xrightarrow[{[(-2)\mathbf{3}+\mathbf{4}]}]{\tau} \left[\begin{array}{cccc} 1 & \mathbf{0} & 0 & \mathbf{0}\\ 2 & \mathbf{0} & 2 & \mathbf{0}\\ 3 & \mathbf{0} & 2 & \mathbf{0}\\\hline 1 & -2 & -2 & 2\\ & 1 & & 0\\ & 0 & 1 & -2\\ & 0 & & 1\end{array}\right] = \left[\begin{array}{c}\mathbf{K}\\\hline \mathbf{E}\end{array}\right]$$

If $\mathbf{A}\left(\mathbf{E}_{|j}\right) = \mathbf{0}$ them $\mathbf{E}_{|j}$ is a solution to $\mathbf{A}\boldsymbol{x} = \mathbf{0}$

The number of pivots of $\mathbf{K}$ is the *rank* of a matrix

F17

(**Lecture** 7) S-8 How to compute $\mathcal{N}(\mathbf{A})$: general solution

The general solution: $\mathcal{N}(\mathbf{A})$

What is the set of ALL solutions?

- $\mathcal{N}(\mathbf{A}) = \Big\{ \qquad\qquad \Big\}$
- $\mathcal{N}(\mathbf{A}) = \left\{ \boldsymbol{x} \in \qquad\qquad \right\}$

How many special solutions are there?

How many null columns are there?

F18

(**Lecture** 7) S-9 Why aren't there more solutions?

Consider $\mathbf{A}\boldsymbol{x} = \mathbf{0}$ and $\mathbf{AE} = \mathbf{K}$ $\qquad$ ($\mathbf{E} = \mathbf{I}_{\tau_1\cdots\tau_k}$ fullrank)

Is $\boldsymbol{x}$ a combination of cols. of $\mathbf{E}$? $\quad (\boldsymbol{x} = \mathbf{E}\boldsymbol{y})$

Using $\boldsymbol{y} = \mathbf{E}^{-1}\boldsymbol{x}$, *we have that* $\boldsymbol{x} = \mathbf{E}\boldsymbol{y}$

Do we need all columns of $\mathbf{E}$ *to get* $\boldsymbol{x}$*?*

$$\left[\begin{array}{c}\mathbf{A}\\\hline \mathbf{I}\end{array}\right] \to \left[\begin{array}{c}\mathbf{K}\\\hline \mathbf{E}\end{array}\right] = \left[\begin{array}{ccccc} * & 0 & 0 & 0 & 0\\ * & * & 0 & 0 & 0\\ * & * & * & 0 & 0\\\hline \mathbf{E}_{|1} & \mathbf{E}_{|2} & \mathbf{E}_{|3} & \mathbf{E}_{|4} & \mathbf{E}_{|5}\end{array}\right]$$

$$\mathbf{A}\boldsymbol{x} = \mathbf{AE}\boldsymbol{y} = \mathbf{K}\boldsymbol{y} = \mathbf{0} \;\Rightarrow\; (y_j = ?\ \text{for pivot columns})$$

$\forall \boldsymbol{x} \in \mathcal{N}(\mathbf{A})$, $\boldsymbol{x}$ *is a combination of the* special solutions

F19

(**Lecture** 7) S-10 Computing $\mathcal{N}(\mathbf{A})$: complete algorithm for solving $\mathbf{A}\boldsymbol{x} = 0$

An algorithm for solving $\mathbf{A}\boldsymbol{x} = \mathbf{0}$

1. Find a pre-echelon form: $\left[\begin{array}{c}\mathbf{A}\\ \hline \mathbf{I}\end{array}\right] \xrightarrow{\tau_1\cdots\tau_k} \left[\begin{array}{c}\mathbf{K}\\ \hline \mathbf{E}\end{array}\right]$

2. If there are *special solutions:*
   - Complete solution
$$\mathcal{N}(\mathbf{A}) = \{\text{linear combinations of special solutions }\}$$

3. If no *special solutions*
   - Complete solution:
$$\mathcal{N}(\mathbf{A}) = \{\mathbf{0}\}$$

F20

(**Lecture** 7) S-11 Another example: $\mathcal{N}(\mathbf{A}^\intercal)$

$$\left[\begin{array}{c}\mathbf{A}^\intercal\\ \hline \mathbf{I}\end{array}\right] = \left[\begin{array}{ccc}1&2&3\\2&4&6\\2&6&8\\2&8&10\\ \hline 1&0&0\\0&1&0\\0&0&1\end{array}\right] \xrightarrow[{[(-1)\mathbf{2}+\mathbf{3}]}]{\substack{\tau\\ {[(-2)\mathbf{1}+\mathbf{2}]}\\ {[(-3)\mathbf{1}+\mathbf{3}]}}} \qquad = \left[\begin{array}{c}\mathbf{L}\\ \hline \mathbf{E}\end{array}\right]$$

How many pivots are there?
How many free columns How many special solutions
set of solutions to $\mathbf{A}^\intercal\boldsymbol{x} = \mathbf{0}$?

F21

The lecture ends here

Questions of the Lecture 7

(L-7) Question 1. Reduce the matrices to a pre-echelon form, to find their ranks. Describe the nullspace with parametric equations.

(a) $\mathbf{A} = \begin{bmatrix} 1 & 2 & 0 & 5\\ 2 & 3 & 1 & 4\\ -1 & -1 & -1 & 1\end{bmatrix}$.

(b) $\mathbf{F} = \begin{bmatrix} 1 & 2 & 1\\ -1 & 3 & 4\\ 2 & -1 & -3\end{bmatrix}$

(c) $\mathbf{G} = \begin{bmatrix} 1 & 2 & 1\\ 0 & 3 & 1\\ -2 & -1 & 4\end{bmatrix}$.

(d) $\mathbf{H} = \begin{bmatrix} 1 & 3\\ 2 & 1\\ -1 & -3\end{bmatrix}$.

(L-7) QUESTION 2. Describe the nullspace of the matrices with parametric equations

(a) $\mathbf{A} = \begin{bmatrix} 1 & 2 & -2 \\ 2 & 1 & 0 \end{bmatrix}$.

(b) $\mathbf{F} = \begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix}$.

(c) $\mathbf{G} = \begin{bmatrix} 1 & 2 & -4 \\ -1 & 1 & 3 \\ 1 & 5 & -5 \end{bmatrix}$.

(L-7) QUESTION 3. Reduce $\mathbf{A} = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{bmatrix}$ to pre-echelon form, to find their ranks. Find the special solutions to $\mathbf{A}\boldsymbol{x} = \mathbf{0}$. Find all solutions.
(Strang, 2006, exercise 2 from section 2.2.)

(L-7) QUESTION 4. Find a pre-echelon form and the rank of these matrices and the complete solution to the systems $\mathbf{A}\boldsymbol{x} = \mathbf{0}$:

(a) The 3 by 4 matrix of all ones.
(b) The 4 by 4 matrix with $a_{ij} = (-1)^{ij}$.
(c) The 3 by 4 matrix with $a_{ij} = (-1)^{j}$.

(Strang, 2006, exercise 13 from section 2.2.)

(L-7) QUESTION 5. The matrix $\mathbf{A}$ has two special solutions:

$$\boldsymbol{x}_1 = \begin{pmatrix} c \\ 1 \\ 0 \end{pmatrix}; \qquad \boldsymbol{x}_2 = \begin{pmatrix} d \\ 0 \\ 1 \end{pmatrix}$$

(a) Describe all the possibilities for the number of columns of $\mathbf{A}$.
(b) Describe all the possibilities for the number of rows of $\mathbf{A}$.
(c) Describe all the possibilities for the rank of $\mathbf{A}$.

Briefly explain your answers.
(*MIT Course 18.06 Quiz 1, Fall, 2008*)

(L-7) QUESTION 6. Suppose $\mathbf{A}$ has column reduced echelon form $\mathbf{R}$

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & \clubsuit \\ 2 & a & \clubsuit \\ 1 & 1 & \clubsuit \\ b & 8 & \clubsuit \end{bmatrix}, \quad \mathbf{R} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 2 & 0 \end{bmatrix}.$$

(a) What can you say about column 3 of $\mathbf{A}$?
(b) What are the numbers a and b?
(c) Describe the nullspace of $\mathbf{A}$ if: $\quad \mathbf{A} \begin{bmatrix} -1 & 2 & 1 \\ 1 & -1 & -2 \\ 0 & 0 & 1 \end{bmatrix} = \mathbf{R}$.

Based on MIT Course 18.06 Quiz 1, March 1, 2004

(L-7) QUESTION 7. Find the reduced column echelon form of these matrices

(a) $\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 1 & 2 & 3 \end{bmatrix}$.

(b) $\mathbf{B} = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 2 & 2 \\ 1 & 0 & 1 \end{bmatrix}$.

(c) $\mathbf{C} = \begin{bmatrix} 1 & 2 & 1 & 2 & 1 \\ 2 & 1 & 2 & 1 & 2 \\ 0 & 1 & 0 & 1 & 0 \end{bmatrix}$.

(d) $\mathbf{D} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \end{bmatrix}$.

(L-7) QUESTION 8. Consider the *invertible* matrix $\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

(a) Knowing $\mathbf{A}$ is invertible, and without any calculation ¿what is its reduced echelon form?
(b) Compute $\mathbf{A}^{-1}$.

(L-7) QUESTION 9. Consider the invertible matrix $\mathbf{A} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

(a) Without any computation, say what is its reduced echelon form.
(b) Find the inverse of $\mathbf{A}$.

End of Questions of the Lecture 7

LECTURE 8: Solving $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$

Lecture 8

(**Lecture** 8) S-1 Highlights of Lesson 8

Highlights of *Lesson 8*

- The column space of a matrix $\mathbf{A}$: solving $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$
- We will completely solve the *general* linear system $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$, . . . *if it has a solution*
  - is there only one solution?
  - or is there a whole family of solutions?

$$\left\{ \boldsymbol{x} = \boldsymbol{x}_p + \boldsymbol{x}_n \;\middle|\; \begin{array}{l} \mathbf{A}(\boldsymbol{x}_p) = \boldsymbol{b} \\ \\ \mathbf{A}(\boldsymbol{x}_n) = \mathbf{0} \end{array} \right\}$$

F22

(**Lecture** 8) S-2 The subspaces of a matrix: the column space $\mathcal{C}(\mathbf{A})$

$$\mathbf{A} = \begin{bmatrix} 1 & 3 \\ 2 & 3 \\ 4 & 1 \end{bmatrix}$$

These column vectors are in

How do I fill the set out to be a subspace?. . . Taking. . .

This subspace is called *column space of* $\mathbf{A}$: $\mathcal{C}(\mathbf{A})$.

So $\mathcal{C}(\mathbf{A})$ is a subspace of

F23

(**Lecture** 8) S-3 The subspaces of a matrix: The column space $\mathcal{C}(\mathbf{A})$

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 2 & 2 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 8 & 10 \end{bmatrix}$$

$\mathcal{C}(\mathbf{A})$ is a subspace of

What is in $\mathcal{C}(\mathbf{A})$?

Is that space $\mathcal{C}(\mathbf{A})$ the whole three dimensional $\mathbb{R}^3$ space?

Let's connect this question with linear equations. . .

F24

(**Lecture** 8) S-**4** **Link between $\mathcal{C}(\mathbf{A})$ and $\mathbf{A}x = b$**

does $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$ have a solution for every $\boldsymbol{b}$? *(the question for today)*

$$\mathbf{A}\boldsymbol{x} = \begin{bmatrix} 1 & 2 & 2 & 2 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 8 & 10 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

Which right-hand sides allow me to solve this?

Can I find a solution for $\boldsymbol{b}_1 = (1,\ 2,\ 3,)$? and $\boldsymbol{b}_2 = (2,\ 6,\ 8,)$? and $\boldsymbol{b}_3 = (0,\ 0,\ 0,)$? and $\boldsymbol{b}_4 = (3,\ 6,\ 9,)$? and $\boldsymbol{b}_5 = (1,\ 0,\ 0,)$?

F25

(**Lecture** 8) S-**5** **Link between $\mathcal{C}(\mathbf{A})$ and $\mathbf{A}x = b$**

$$\mathbf{A}\boldsymbol{x} = \begin{bmatrix} 1 & 2 & 2 & 2 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 8 & 10 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

Can I throw away any column and keep the same? $\mathcal{C}(\mathbf{A})$?

and elimination will show which columns are linear combination of those to the left of it.

But how does Gaussian elimination affect $\mathcal{N}(\mathbf{A})$ and $\mathcal{C}(\mathbf{A})$?

F26

(**Lecture** 8) S-**6** **In the next example we will use the reduced echelon form**

Gauss-Jordan elimination: *all pivots are 1, with zeros at the left*

$$\left[\begin{array}{c} \mathbf{A} \\ \hline \mathbf{I} \end{array}\right] = \left[\begin{array}{cccc} 1 & 2 & 2 & 2 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 8 & 10 \\ \hline 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow[\substack{[(-2)\mathbf{1}+\mathbf{2}] \\ [(-2)\mathbf{1}+\mathbf{3}] \\ [(-2)\mathbf{1}+\mathbf{4}] \\ [(-2)\mathbf{3}+\mathbf{4}] \\ [\mathbf{2}\rightleftharpoons\mathbf{3}]}]{\tau} \left[\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 2 & 2 & 0 & 0 \\ 3 & 2 & 0 & 0 \\ \hline 1 & -2 & -2 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow[\substack{[(-1)\mathbf{2}+\mathbf{1}] \\ [(\frac{1}{2})\mathbf{2}]}]{\tau} \left[\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ \hline 3 & -1 & -2 & 2 \\ 0 & 0 & 1 & 0 \\ -1 & \frac{1}{2} & 0 & -2 \\ 0 & 0 & 0 & 1 \end{array}\right] = \left[\begin{array}{c} \mathbf{R} \\ \hline \mathbf{E} \end{array}\right]$$

$$\boxed{\begin{aligned} \left(\mathbf{A}_{\tau_1\cdots\tau_k}\right)_{|j} &= \mathbf{A}\left(\mathbf{I}_{\tau_1\cdots\tau_k}\right)_{|j} \\ \left(\mathbf{A}_{\tau_1\cdots\tau_k}\right)\left(\mathbf{I}_{\tau_k^{-1}\cdots\tau_1^{-1}}\right)_{|j} &= \mathbf{A}_{|j} \end{aligned}} \implies \mathcal{C}(\mathbf{A}) = \mathcal{C}\left(\mathbf{A}_{\tau_1\cdots\tau_k}\right).$$

But in general $\mathcal{N}(\mathbf{A}) \neq \mathcal{N}\left(\mathbf{A}_{\tau_1\cdots\tau_k}\right)$.

F27

(**Lecture** 8) S-7 Linear system of equations example

$$\begin{cases} x_1 + 2x_2 + 2x_3 + \ \ 2x_4 = b_1 \\ 2x_1 + 4x_2 + 6x_3 + \ \ 8x_4 = b_2 \\ 3x_1 + 6x_2 + 8x_3 + 10x_4 = b_3 \end{cases}$$

What is going to discover elimination about the columns?

What must $(b_1, b_2, b_3,)$ fulfil for a solution to exist?
If $b_1 = 1$ and $b_2 = 5$, what is the only b_3 that would be OK?
Let's see!

$$\mathbf{A}\boldsymbol{x} = \boldsymbol{b} \quad \Leftrightarrow \quad \mathbf{A}\boldsymbol{x} - 1\boldsymbol{b} = \mathbf{0} \quad \Leftrightarrow \quad \left[\ \mathbf{A} \mid -\boldsymbol{b}\ \right] \begin{pmatrix} \boldsymbol{x} \\ 1 \end{pmatrix} = \mathbf{0}$$

F28

(**Lecture** 8) S-8 Linear system of equations: condition for solvability

$$\left[\ \mathbf{A} \mid -\boldsymbol{b}\ \right] (\boldsymbol{x},\quad 1,) = \mathbf{0}$$

$$\left[\begin{array}{c|c} \mathbf{A} & -\boldsymbol{b} \\ \hline \mathbf{I} & \mathbf{0} \\ & 1 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & -b_1 \\ 0 & 1 & 0 & 0 & -b_2 \\ 1 & 1 & 0 & 0 & -b_3 \\ \hline 3 & -1 & -2 & 2 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ -1 & \frac{1}{2} & 0 & -2 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow[{[(b_2)2+5]}]{\substack{\tau \\ [(b_1)1+5]}} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & b_1+b_2-b_3 \\ \hline 3 & -1 & -2 & 2 & 3b_1-b_2 \\ 0 & 0 & 1 & 0 & 0 \\ -1 & \frac{1}{2} & 0 & -2 & -b_1+\frac{1}{2}b_2 \\ 0 & 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 \end{array}\right] = \left[\begin{array}{c|c} \mathbf{R} & \mathbf{0} \\ \hline \mathbf{E} & \boldsymbol{x}_p \\ & 1 \end{array}\right]$$

Then the condition for solvability is:

If $b_1 = 1$ and $b_2 = 5$ then $b_3 =$
If $\boldsymbol{b} = (1,\quad 5,\quad 6,)$ what is the last column when the system is solvable?
Solve for $\boldsymbol{b} = (2,\quad 2,\quad 4,)$

F29

Let's check:

$$\mathbf{A}\boldsymbol{x}_p = (3b_1 - b_2)\begin{pmatrix}1\\2\\3\end{pmatrix} + 0\begin{pmatrix}2\\4\\6\end{pmatrix} + \left(-b_1 + \frac{1}{2}b_2\right)\begin{pmatrix}2\\6\\8\end{pmatrix} + 0\begin{pmatrix}2\\8\\10\end{pmatrix} = \begin{pmatrix}b_1\\b_2\\b_1+b_2\end{pmatrix};$$

the system is solvable if $b_1 + b_2 = b_3$. If $\boldsymbol{b} = \begin{pmatrix}1\\5\\6\end{pmatrix}$, then

$$\mathbf{A}\boldsymbol{x}_p = -2\begin{pmatrix}1\\2\\3\end{pmatrix} + 0\begin{pmatrix}2\\4\\6\end{pmatrix} + \frac{3}{2}\begin{pmatrix}2\\6\\8\end{pmatrix} + 0\begin{pmatrix}2\\8\\10\end{pmatrix} = \begin{pmatrix}-2\\-4\\-6\end{pmatrix} + \begin{pmatrix}3\\9\\12\end{pmatrix} = \begin{pmatrix}1\\5\\6\end{pmatrix} = \boldsymbol{b}.$$

Try with another $\boldsymbol{b}$!

```
b1,b2,b3 = sympy.symbols('b1 b2 b3')
A  = Matrix([[1,2,2,2],[2,4,6,8],[3,6,8,10]])
b  = Vector([b1,b2,b3])
SEL(A, b, 1)
SEL(A, b.subs(b1,2).subs(b2,2).subs(b3,4), 1)
```

(**Lecture** 8) S-9 An algorithm to solve the linear system $\mathbf{A}\boldsymbol{x} = b$

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 2 & 2 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 8 & 10 \end{bmatrix}; \quad \boldsymbol{b} = \begin{pmatrix} 2 \\ 2 \\ 4 \end{pmatrix}; \qquad \mathbf{A}\boldsymbol{x} = \boldsymbol{b}$$

$$\left[\begin{array}{cccc|c} 1 & 2 & 2 & 2 & -2 \\ 2 & 4 & 6 & 8 & -2 \\ 3 & 6 & 8 & 10 & -4 \\ \hline 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 2 & 0 & 2 & 4 & 2 \\ 3 & 0 & 2 & 4 & 2 \\ \hline 1 & -2 & -2 & -2 & 2 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow[{[(-1)\mathbf{3}+\mathbf{5}]}]{\substack{\tau \\ {[(-2)\mathbf{3}+\mathbf{4}]}}} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 2 & 0 & 2 & 0 & 0 \\ 3 & 0 & 2 & 0 & 0 \\ \hline 1 & -2 & -2 & 2 & 4 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -2 & -1 \\ 0 & 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\left\{ \boldsymbol{x} \in \mathbb{R}^4 \;\middle|\; \text{exists} \begin{pmatrix} a \\ b \end{pmatrix} \in \mathbb{R}^2 \text{ such that } \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \\ -1 \\ 0 \end{pmatrix} + a \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + b \begin{pmatrix} 2 \\ 0 \\ -2 \\ 1 \end{pmatrix} \right\}$$

F30

Another expression for the solution set

$$\left\{ \boldsymbol{v} \in \mathbb{R}^4 \;\middle|\; \exists \boldsymbol{p} \in \mathbb{R}^2, \ \boldsymbol{v} = \begin{pmatrix} 4 \\ 0 \\ -1 \\ 0 \end{pmatrix} + \begin{bmatrix} -2 & 2 \\ 1 & 0 \\ 0 & -2 \\ 0 & 1 \end{bmatrix} \boldsymbol{p} \right\}$$

(**Lecture** 8) S-10 Complete algorithm to solve the linear system $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$

Apply elimination to solve $\left[\ \mathbf{A} \mid -\boldsymbol{b}\ \right] \begin{pmatrix} \boldsymbol{x} \\ 1 \end{pmatrix} = \mathbf{0}$

$$\left[\begin{array}{c|c} \mathbf{A} & -\boldsymbol{b} \\ \hline \mathbf{I} & \mathbf{0} \\ \hline 0\cdots0 & 1 \end{array}\right] \xrightarrow{\text{Elimination}} \left[\begin{array}{c|c} \mathbf{K} & \boldsymbol{c} \\ \hline \mathbf{E} & \boldsymbol{x}_p \\ \hline 0\cdots0 & 1 \end{array}\right], \quad \text{where} \quad \mathbf{K} = \mathbf{AE}.$$

- If $\boldsymbol{c} \neq \mathbf{0}$, the system $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$ is not solvable.
- If $\boldsymbol{c} = \mathbf{0}$ then $\boldsymbol{b} \in \mathcal{C}(\mathbf{A})$ and the set of solutions is

$$\left\{ \boldsymbol{x} \in \mathbb{R}^n \;\middle|\; \text{exists } \boldsymbol{y} \in \mathcal{N}(\mathbf{A}) \text{ such that } \boldsymbol{x} = \boldsymbol{x}_p + \boldsymbol{y} \right\}.$$

If $\mathcal{N}(\mathbf{A}) = \{\mathbf{0}\}$, then $\boldsymbol{x}_p$ is the unique solution.

F31

Note that

$$\begin{cases} \mathbf{A}\boldsymbol{x} & = \boldsymbol{b} \\ \mathbf{A}\boldsymbol{y} & = \mathbf{0} \end{cases} \qquad \Rightarrow \qquad \mathbf{A}(\boldsymbol{x} + \boldsymbol{y}) = \boldsymbol{b}$$

So

$$(\boldsymbol{x} + \boldsymbol{y}) \quad \text{is another solution to } \mathbf{A}\boldsymbol{x} = \boldsymbol{b}.$$

Also note that

$$\begin{cases} \mathbf{A}\boldsymbol{x} & = \boldsymbol{b} \\ \mathbf{A}\boldsymbol{y} & = \boldsymbol{b} \end{cases} \qquad \Rightarrow \qquad \mathbf{A}(\boldsymbol{x} - \boldsymbol{y}) = \mathbf{0}$$

So, if $\boldsymbol{x}$ and $\boldsymbol{y}$ are solutions to $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$ then

$$(\boldsymbol{x} - \boldsymbol{y}) \in \mathcal{N}(\mathbf{A}).$$

(**Lecture** 8) S-11 **Rouché-Frobenius theorem**

Sist. $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$	$r = m = n$	$r = n < m$	$r = m < n$	$r < m;\quad r < n$
solutions				

$$\left[\begin{array}{c|c} \mathbf{A} & \text{-}\boldsymbol{b} \\ \hline \mathbf{I} & \mathbf{0} \\ \hline \mathbf{0} & 1 \end{array}\right] \xrightarrow{\tau_1 \cdots \tau_k} \left[\begin{array}{c|c} \mathbf{R} & \text{-}\boldsymbol{b} \\ \hline \mathbf{E} & \mathbf{0} \\ \hline \mathbf{0} & 1 \end{array}\right] = \left[\begin{array}{ccccccc|c} \mathbf{1} & 0 & \cdots & 0 & \cdots & 0 & 0 & \text{-}b_1 \\ 0 & \mathbf{1} & \cdots & 0 & \cdots & 0 & 0 & \text{-}b_2 \\ 0 & 0 & & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \mathbf{1} & \cdots & 0 & 0 & \text{-}b_k \\ \vdots & \vdots & \cdots & \vdots & \cdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & & \vdots & & \vdots & \vdots & \text{-}b_m \\ \hline \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \hline 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & 1 \end{array}\right]$$

where $\mathbf{A}$ *(with order $m \times n$) has rank r*; and where "**1**" are pivots.

$$\boxed{\text{Ranks of } \mathbf{A} \quad \text{and} \quad \left[\mathbf{A} \,|\, -\boldsymbol{b}\right] \quad \text{tells you everything about the number of solutions}}$$

F32

The lecture ends here

Questions of the Lecture 8

(L-8) Question 1. Which of these rules give a correct definition of the rank of $\mathbf{A}$?

(a) The number of non-zero columns in $\mathbf{R}$ (reduced column echelon form).
(b) The number of columns minus the total number of rows.
(c) The number of columns minus the number of free columns.
(d) The number of ones in $\mathbf{R}$.

(Strang, 2006, exercise 12 from section 2.2.)

(L-8) Question 2. Find the complete solution (also called the *general solution*) to

$$\begin{bmatrix} 1 & 3 & 1 & 2 \\ 2 & 6 & 4 & 8 \\ 0 & 0 & 2 & 4 \end{bmatrix} \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix}$$

(Strang, 2003, exercise 4 from section 3.4.)

(L-8) Question 3. Find the complete solution (also called the *general solution*) to

$$\begin{bmatrix} 1 & 2 & -1 & -2 & 1 \\ 1 & 2 & 0 & 0 & 3 \\ 2 & 4 & 1 & 2 & 9 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ -4 \end{pmatrix}$$

(L-8) Question 4. Resuelva el siguiente sistema de ecuaciones

$$\begin{aligned} x_1 + x_3 + x_5 &= 1 \\ x_2 + x_4 &= 1 \\ x_1 + x_2 + x_3 + x_4 &= 2 \\ x_3 + x_4 &= 2 \end{aligned}$$

(L-8) Question 5.

Consider

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 3 \\ 0 & 2 & 0 & 6 \end{bmatrix}; \qquad \boldsymbol{b} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}.$$

(a) Find the column echelon form
(b) Find the free variables
(c) Find the special solutions:
(d) $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$ is consistent (has a solution) when $\boldsymbol{b}$ satisfies $b_2 =$ ____.
(e) Find the complete solution to the system when b_2 satisfies the consistency condition.

(Strang, 2006, exercise 3 from section 2.2.)

(L-8) Question 6. Carry out the same steps as in the previous problem to find the complete solution of $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$.

$$\mathbf{A} = \begin{bmatrix} 0 & 0 \\ 1 & 2 \\ 0 & 0 \\ 3 & 6 \end{bmatrix}; \quad \boldsymbol{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{pmatrix}.$$

(Strang, 2006, exercise 4 from section 2.2.)

(L-8) Question 7. Describe the set of attainable right-hand sides $\boldsymbol{b}$? (the column space $\mathcal{C}(\mathbf{A})$) for

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 2 & 3 \end{bmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix},$$

by finding the constraints on $\boldsymbol{b}$ (after elimination). What is the rank? give a particular solution to the system? (Strang, 2006, exercise 6 from section 2.2.)

(L-8) Question 8. Suppose a paint company paints automobiles, trains and planes.

Each automobile takes 10 man hours to prepare, 30 man hours to paint, and 12 man hours to add finishing touches (the painters are quite meticulous).

Each train takes 20 man hours to prepare, 75 man hours to paint, and 36 man hours to add finishing touches.

Each plane takes 40 man hours to prepare, 135 man hours to paint, and 64 man hours to add finishing touches.

If the paint company decides to use 760 man hours towards preparation, 2595 man hours towards painting, and 1224 man hours towards finishing touches each week, how many planes, trains and automobiles do they paint each week?

(L-8) Question 9. Para el sistema $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$ dado por

$$\begin{bmatrix} 1 & 0 & 4 \\ 2 & 1 & 10 \\ 3 & 1 & c \end{bmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 6 \\ 14 \\ 20 \end{pmatrix}$$

(a) Encuentre el valor de c que hace a la matriz $\mathbf{A}$ no invertible. Use dicho valor en los apartados siguientes.
(b) Encuentre la solución completa al sistema $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$.

(c) Describa el sistema de ecuaciones mediante la visión por columnas (columnas de $\mathbf{A}$ y el vector $\boldsymbol{b}$), o bien mediante la visión por filas (las tres ecuaciones del sistema).

(L-8) QUESTION 10. Construct a matrix whose column space contains $(1,1,5)$ and $(0,3,1)$ and whose nullspace contains $(1,1,2)$.
(Strang, 2006, exercise 62 from section 2.2.)

(L-8) QUESTION 11. For which vectors $\boldsymbol{b}$ do these systems have a solution?

(a) $\begin{bmatrix}1&1&1\\0&1&1\\0&0&1\end{bmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=\begin{pmatrix}b_1\\b_2\\b_3\end{pmatrix}$ (b) $\begin{bmatrix}1&1&1\\0&1&1\\0&0&0\end{bmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=\begin{pmatrix}b_1\\b_2\\b_3\end{pmatrix}$

(Strang, 2006, exercise 24 from section 2.1.)

(L-8) QUESTION 12. Under what conditions on b_1 and b_2 (if any) does $\mathbf{A}\boldsymbol{x}=\boldsymbol{b}$ have a solution?

$$\mathbf{A}=\begin{bmatrix}1&2&0&3\\2&4&0&7\end{bmatrix},\qquad \boldsymbol{b}=\begin{pmatrix}b_1\\b_2\end{pmatrix}.$$

Find two vectors in the nullspace of $\mathbf{A}$, and the complete solution to $\mathbf{A}\boldsymbol{x}=\boldsymbol{b}$.
(Strang, 2006, exercise 8 from section 2.2.)

(L-8) QUESTION 13. Sea la matriz

$$\underset{3\times 4}{\mathbf{B}}=\begin{bmatrix}1&0&0&0\\0&1&0&0\\1&-1&0&0\end{bmatrix}\begin{bmatrix}1&1&-0&1\\0&1&1&-1\\0&0&1&-0\\0&0&0&1\end{bmatrix}$$

(a) Sin realizar la multiplicación, diga una base de $\mathcal{N}\left(\mathbf{B}\right)$, y el rango de $\mathbf{B}$. Explique su respuesta.

(b) ¿Cuál es la solución completa a $\mathbf{B}\boldsymbol{x}=\begin{pmatrix}1\\0\\1\end{pmatrix}$?

(L-8) QUESTION 14. For which right-hand sides (find a condition on $\boldsymbol{b}$) are these systems solvable?

(a)

$$\begin{bmatrix}1&4&2\\2&8&4\\-1&-4&-2\end{bmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=\begin{pmatrix}b_1\\b_2\\b_3\end{pmatrix}$$

Is the column space $\mathcal{C}\left(\mathbf{A}\right)$ the whole 3 dimensional space $\mathbb{R}^3$? , or is it only a plane? a line? a point?

(b)

$$\begin{bmatrix}1&4\\2&9\\-1&-4\end{bmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix}=\begin{pmatrix}b_1\\b_2\\b_3\end{pmatrix}$$

Is the column space $\mathcal{C}\left(\mathbf{A}\right)$ the whole 3 dimensional space $\mathbb{R}^3$? , or is it only a plane? a line? a point? *Based on (Strang, 2006, exercise 22 from section 2.1.)*

(L-8) QUESTION 15. The complete solution to $\mathbf{A}\boldsymbol{x}=\boldsymbol{b}\in\mathbb{R}^m$ is:

$$\left\{\boldsymbol{x}\in\mathbb{R}^3\;\middle|\;\exists\, c_1,c_2\in\mathbb{R}\text{ such that }\boldsymbol{x}=\begin{pmatrix}1\\0\\0\end{pmatrix}+c_1\begin{pmatrix}0\\1\\0\end{pmatrix}+c_2\begin{pmatrix}0\\0\\1\end{pmatrix}\right\}.\qquad\text{What is }\mathbf{A}?$$

(L-8) QUESTION 16. Mike, Shai, and Tara all decide that they are unhappy with the color scheme at Math II classroom, and they do something about it. They go down to the paint store, and each buy some paint. Mike buys one gallon of red paint, six gallons of blue paint, and one gallon of yellow paint. He spends 44 euros. Shai, on the other hand, buys no red paint, two gallons of blue paint and three gallons of yellow paint. He spends 24 euros. Tara finally buys one gallon of red paint and five gallons of blue paint, and spends 33 euros.

(a) How much does each color of paint cost?
(b) What is wrong with your answer to the previous problem?
(c) When Mike, Shai and Tara compare receipts, they realize that one of them was charged 4 too little. Who was it?
(d) Tras intentar dar respuesta a la pregunta anterior, se habrá dado cuenta de que es un tanto "trabajoso" dar con el resultado. Intente lo siguiente: genere la matriz ampliada $\begin{bmatrix}\mathbf{A}|\boldsymbol{a} & \boldsymbol{b} & \boldsymbol{c}\end{bmatrix}$ donde $\mathbf{A}$ es la matriz de coeficientes del sistema de ecuaciones, y $\boldsymbol{a}$ es el vector de precios suponiendo que a Ana deberían haberle cobrado 4 euros más (es decir 48 en lugar de 44), $\boldsymbol{b}$ el vector de precios suponiendo que sólo a Belén deberían haberle cobrado 4 euros más, y $\boldsymbol{c}$ lo mismo para Carlos. Calcule la forma escalonada reducida de la matriz ampliada. A la vista de lo obtenido ¿cuanto vale cada bote de pintura? y ¿a quien han cobrado 4 euros de menos?

(L-8) QUESTION 17. Suponga que el sistema de ecuaciones $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$ es consistente (que tiene solución), donde $\underset{m\times n}{\mathbf{A}}$ y $\boldsymbol{x} = (x_1, ..., x_n)$. **Demuestre** las siguientes afirmaciones:
(a) $\boldsymbol{b} \in \mathcal{C}\left(\mathbf{A}\right)$.
(b) Si $\boldsymbol{x}_0$ es una solución particular del sistema, entonces cualquier vector de la forma $\boldsymbol{x}_0 + \boldsymbol{z}$, donde $\boldsymbol{z} \in \mathcal{N}\left(\mathbf{A}\right)$, es también solución del sistema.
(c) Demuestre que si hay dependencia lineal entre las columnas de $\mathbf{A}$, entonces hay más de una solución.

(L-8) QUESTION 18. Solve the following system of equations using Gaussian elimination.

$$\begin{cases} 3x & + & y & + & z & = & 6 \\ x & - & y & - & z & = & -2 \\ & & 4y & + & z & = & 3 \end{cases}$$

(L-8) QUESTION 19. Write these ancient problems in a 2 by 2 matrix form $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$, and solve them:
(a) X is twice as old as Y and their ages add to 39.
(b) $(x, y,) = (2, 5,)$ and $(x, y,) = (3, 7,)$ lie on the line $y = mx + c$. Find m and c.
(Strang, 1988, exercise 32 from section 1.4.)

(L-8) QUESTION 20. The parabola $y = a + bx + cx^2$ goes through the points $(x, y,) = (1, 4,)$, $(2, 8,)$ and $(3, 14,)$. Find and solve a matrix equation for the unknowns $\boldsymbol{x} = \begin{pmatrix}a, & b, & c,\end{pmatrix}$.
(Strang, 1988, exercise 33 from section 1.4.)

(L-8) QUESTION 21. Explain why the system

$$\begin{cases} u & + & v & + & w & = & 2 \\ u & + & 2v & + & 3w & = & 1 \\ & & v & + & 2w & = & 0 \end{cases}$$

is singular and has no solution.
What value should replace the last zero on the right hand side, to allow the the equations to have solutions— and what is one of the solutions?
(Strang, 1988, exercise 8 from section 1.2.)

(L-8) QUESTION 22. Choose a coefficient b that makes this system singular. Then choose a value for g that makes it solvable. Find two solutions in that singular case.

$$\begin{cases} 2x + by & = 16 \\ 4x + 8y & = g \end{cases}$$

Based on (Strang, 2003, exercise 6 from section 2.2.)

(L-8) QUESTION 23. Solve the following nonsingular triangular system. Show that your solution gives the linear combination of the columns that equals the column of the right $\boldsymbol{b} = \begin{pmatrix}b_1, & b_2, & b_3,\end{pmatrix}$.

$$\begin{aligned} u - v + w &= b_1 \\ v + w &= b_2 \\ w &= b_3. \end{aligned}$$

Check your answer multiplying $\mathbf{A}$ by your solution vector.
(Strang, 1988, exercise 2 from section 1.2.)

(L-8) QUESTION 24. Find $\mathbf{A}$ and $\mathbf{B}$ with the given property or explain why you can't.

(a) The only solution to $\mathbf{A}\boldsymbol{x} = \begin{pmatrix}1\\2\\3\end{pmatrix}$ is $\boldsymbol{x} = \begin{pmatrix}0\\1\end{pmatrix}$.

(b) The only solution to $\mathbf{B}\boldsymbol{x} = \begin{pmatrix}0\\1\end{pmatrix}$ is $\boldsymbol{x} = \begin{pmatrix}1\\2\\3\end{pmatrix}$.

(Strang, 2006, exercise 49 from section 2.2.)

(L-8) QUESTION 25. The complete solution to $\mathbf{A}x = \begin{pmatrix}1\\3\end{pmatrix}$ is $\boldsymbol{x} = \begin{pmatrix}1\\0\end{pmatrix} + c\begin{pmatrix}0\\1\end{pmatrix}$. Find $\mathbf{A}$.
(Strang, 2006, exercise 50 from section 2.2.)

(L-8) QUESTION 26. Suppose the fifth column of $\mathbf{L}$ has no pivot. Then x_5 is a ____________ variable. The zero vector (is) (is not) the only solution to $\mathbf{A}\boldsymbol{x} = \mathbf{0}$. If $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$ has a solution, then it has____________ solutions.
(Strang, 2006, exercise 40 from section 2.2.)

(L-8) QUESTION 27. Consider a linear system of algebraic equation $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$. Here the matrix $\mathbf{A}$ has three rows and four columns.

(a) Does such a linear system always have at least one solution? If not provide an example for which no solution exists.
(b) Can such a linear system have a unique solution? If so, provide and example of a problem with this property.
(c) Formulate, if possible, necessary and sufficient conditions on $\mathbf{A}$ and $\boldsymbol{b}$ which guarantee that at least one solution exists.
(d) Formulate, if possible, necessary and sufficient conditions on $\mathbf{A}$ which guarantee that at least one solution exists for any choice of $\boldsymbol{b}$.

(L-8) QUESTION 28. By performing column eliminations on the 4×7 matrix $\mathbf{A}$

$$\mathbf{A} = \begin{bmatrix} 1 & -1 & 0 & 2 & 0 & -1 & 0 \\ 2 & -2 & 1 & 5 & 0 & -1 & 0 \\ -3 & 3 & -1 & -7 & 2 & 2 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 1 \end{bmatrix}$$

we got the following matrix $\mathbf{B} = \mathbf{A}_{\boldsymbol{\tau}_1\cdots\boldsymbol{\tau}_k}$:

$$\mathbf{B} = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}; \quad \text{where } \mathbf{I}_{\boldsymbol{\tau}_1\cdots\boldsymbol{\tau}_k} = \begin{bmatrix} 2 & 1 & 0 & -2 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ -4 & 0 & 2 & -1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix}.$$

(a) What is the rank of $\mathbf{A}$? Find the complete solution to $\mathbf{A}\boldsymbol{x} = \mathbf{0}$.
(b) Write, if it is possible, the general solution as a function of x_2, x_4, and x_6.
(c) Is it possible to find a vector $\boldsymbol{b}$ in $\mathbb{R}^4$ that is not in the column space of $\mathbf{A}$ ($\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$ has no solution)? If it is, give an example.
(d) Give a vector $\boldsymbol{b}$ such that the vector $\boldsymbol{x} = \mathbf{I}_{|1}$ is a solution to the system $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$.
(e) If $\boldsymbol{b}$ is the sum of columns of $\mathbf{A}$, find, if it is possible, the full solution to $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$.

A modified version of MIT Course 18.06 Quiz 1, October 4, 2004

(L-8) QUESTION 29. $\underset{m\times n}{\mathbf{A}}$ es una matriz de rango r. Suponga que $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$ no tiene solución para algunos vectores $\boldsymbol{b}$, pero infinitas soluciones para otros vectores $\boldsymbol{b}$.

(a) Decida si el espacio nulo $\mathcal{N}\left(\mathbf{A}\right)$ contiene sólo el vector cero, y explique porqué.
(b) Decida si el espacio columna $\mathcal{C}\left(\mathbf{A}\right)$ es todo $\mathbb{R}^m$ y explique porqué.
(c) Para esta matriz $\mathbf{A}$, encuentre las relaciones entre los números r, m; y entre r y n.
(d) ¿Puede existir un lado derecho $\boldsymbol{b}$ para el que $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$ tenga una y sólo una solución? ¿Porqué es posible o porqué no?

(L-8) QUESTION 30. Sea la matriz

$$\mathbf{A} = \begin{bmatrix} 2 & 3 & 1 & -1 \\ 6 & 9 & 3 & -2 \end{bmatrix}$$

(a) Encuentre un conjunto de soluciones del sistema $\mathbf{A}\boldsymbol{x} = \mathbf{0}$ y describa con él el espacio nulo de $\mathbf{A}$.
(b) Encuentre la solución completa— es decir todas las soluciones (x_1, x_2, x_3, x_4) — de

$$\mathbf{A}\boldsymbol{x} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

(c) Cuando una matriz $\underset{m\times n}{\mathbf{A}}$ tiene rango $r = m$ ¿para qué vectores $\boldsymbol{b}$ el sistema $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$ puede resolverse? ¿Cuantas soluciones especiales tiene $\mathbf{A}\boldsymbol{x} = \mathbf{0}$ (dimensión del espacio nulo)?

(L-8) QUESTION 31. consider the system of linear equations,

$$\left\{\begin{array}{rcrcrcr} x & + & y & + & 2z & = & 1 \\ 2x & + & 2y & - & z & = & 1 \\ & & y & + & cz & = & 2 \end{array}\right.$$

For which number c the system has no solution? Only one? and infinite solutions?

(L-8) QUESTION 32. Consider the following system of linear equations

$$\begin{cases} x - y + 2z & = 1 \\ 2x - 3y + mz & = 3 \\ -x + 2y + 3z & = 2m \end{cases}$$

(a) Show that the system has solution for any value m
(b) Find the solution when $m = -1$.
(c) Is the set of solutions to the system in the last question ($m = -1$) a line in $\mathbb{R}^3$? Is there any m such as the set of solutions to the system is a plane in $\mathbb{R}^3$?... and a point in $\mathbb{R}^3$?
(d) Find the solution to the system when $m = 1$.

(L-8) QUESTION 33. Which descriptions are correct? The solutions $\boldsymbol{x}$ of

$$\mathbf{A}\boldsymbol{x} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 2 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

form:
(a) A plane
(b) A line
(c) A point
(d) A subspace
(e) The nullspace of $\mathbf{A}$.
(f) The column space of $\mathbf{A}$.

(Strang, 2006, exercise 8 from section 2.1.)

(L-8) QUESTION 34. Consider the equation $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$

$$\begin{bmatrix} 1 & 0 \\ 4 & 1 \\ 2 & -1 \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

For which $\boldsymbol{b}$ are there solutions?

Based in *MIT Course 18.06 Quiz 1, Fall 2008*

(L-8) QUESTION 35. En un teatro de barrio, tres grupos están haciendo cola. Hay cuatro tipos de tarifas; tercera edad (t), adulto (a), infantil (i) y tarifa con descuento para empleados del teatro y familiares (d).

El primer grupo compra tres entradas de adulto y tres infantiles por 39 euros.

El segundo grupo compra tres entradas de adulto y cuatro de la tercera edad por 44 euros

El tercer grupo compra dos entradas con descuento y dos entradas infantiles por 22 euros

(a) Si intenta descubrir el precio de cada entrada ¿cuantas soluciones puede encontrar? Ninguna, una, o infinitas

(b) Si las entradas de la tercera edad valen lo mismo que las infantiles. ¿Cuánto vale cada tipo de entrada?

(L-8) QUESTION 36. Consider the following system of linear equations

$$\begin{cases} x_1 + 2x_2 - x_3 + x_4 = -1 \\ -x_1 - 2x_2 + 3x_3 + 5x_4 = -5 \\ -x_1 - 2x_2 - x_3 - 7x_4 = 7 \end{cases}$$

(a) (0.5 pts) What is the rank of the coeficient matrix?

(b) (1.5 pts) Find all solutions to the system of linear equations

(c) (0.5 pts) Describe the geometric shape of the collection of all solutions to the above equations considered as a subset of $\mathbb{R}^4$.

(L-8) QUESTION 37. Consider the following linear system $\mathbf{A}\boldsymbol{x} = \mathbf{0}$ where

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 1 & 2 & 3 \\ a & 1 & 1 & 2 \end{bmatrix}.$$

(a) (0.5^{pts}) Find the values of a such as the set of solutions of the linear system is a line.

(b) (0.5^{pts}) Find the values of a such as the set of solutions of the linear system is a plane?

(L-8) QUESTION 38. Find the complete solution to the system

$$\begin{bmatrix} 1 & 3 & 2 & 4 & -3 \\ 2 & 6 & 0 & -1 & -2 \\ 0 & 0 & 6 & 2 & -1 \\ 1 & 3 & -1 & 4 & 2 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} -7 \\ 0 \\ 12 \\ -6 \end{pmatrix}$$

(L-8) QUESTION 39. Sea la matriz $\underset{3\times 4}{\mathbf{A}}$ y el vector columna $\boldsymbol{b}$ de $\mathbb{R}^3$:

$$\mathbf{A} = \begin{bmatrix} 1 & 3 & 2 & 2 \\ 2 & 7 & 6 & 8 \\ 3 & 9 & 6 & 7 \end{bmatrix}; \qquad \boldsymbol{b} = \begin{pmatrix} 2 \\ 7 \\ 7 \end{pmatrix}$$

(a) Encuentre todas la soluciones al sistema $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$ (si es que existen soluciones). Describa el conjunto de soluciones geométricamente. ¿Es dicho conjunto un sub-espacio vectorial?

(b) ¿Quién es el espacio columna $\mathcal{C}(\mathbf{A})$? Cambie el 7 de la esquina inferior derecha por un número que conduzca a un espacio columna más pequeño de la nueva matriz (digamos $\mathbf{M}$). Dicho número es ____.

(c) Encuentre un lado derecho $\boldsymbol{b}$ tal que, para la nueva matriz, el sistema $\mathbf{M}\boldsymbol{x} = \boldsymbol{b}$ tenga solución; y otro lado derecho $\boldsymbol{b}$ tal que $\mathbf{M}\boldsymbol{x} = \boldsymbol{b}$ no tenga solución.

End of Questions of the Lecture 8

LECTURE 9: Independence, Basis, and Dimension

Lecture 9

(**Lecture** 9) S-**1** Highlights of Lesson 9

Highlights of *Lesson 9*

- Linear *independence*
- vectors *spanning* a space
- **BASIS** and dimension

F33

(**Lecture** 9) S-**2** Homogeneous equations: our starting point

Suppose I have a matrix $\underset{m\times n}{\mathbf{A}}$ with $m < n$ and I look at $\mathbf{A}\boldsymbol{x} = \mathbf{0}$.
(more unknowns than equations ($m < n$), *free columns*)
Then, there's something in $\mathcal{N}(\mathbf{A})$, other than just the zero vector.
There are non-trivial linear combinations $\mathbf{A}\boldsymbol{x}$ that give $\mathbf{0}$

F34

(**Lecture** 9) S-**3** Linear independence

Vectors $\vec{v}_1, \ldots, \vec{v}_n$ are (*linearly*) independent if:
the only linear combination that is equal to $\vec{0}$ is

$$(\vec{v}_1)0 + (\vec{v}_2)0 + \cdots + (\vec{v}_n)0$$

that is

$$\boxed{(\vec{v}_1)p_1 + \cdots + (\vec{v}_n)p_n = \vec{0} \quad \text{only happens when all } p_i \text{ are zero}}$$

$$\boxed{[\vec{v}_1;\ \ldots\ \vec{v}_n;]\boldsymbol{p} = \vec{0} \quad \text{if and only if } \boldsymbol{p} = \mathbf{0}}$$

F35

(**Lecture** 9) S-4 **linear independence: examples in $\mathbb{R}^2$**

Can you find numbers a and b such that $a\boldsymbol{v} + b\boldsymbol{w} = \mathbf{0}$?

- $\boldsymbol{v}$ and $\boldsymbol{w} = 2\boldsymbol{v}$
- $\boldsymbol{v}$ and $\boldsymbol{w} = \mathbf{0}$
- 2 non-aligned vectors
- 3 vectors in $\mathbb{R}^2$

F36

(**Lecture** 9) S-5 **linear independence and rank of a matrix**

Columns of $\underset{m\times n}{\mathbf{A}}$ are:

- independent:

  If the null space $\mathcal{N}\left(\mathbf{A}\right)$ is

- dependent if:

  $\mathbf{A}\boldsymbol{c} = \mathbf{0}$ for some vector $\boldsymbol{c} \neq \mathbf{0}$.

- independent if: $\mathrm{rg}\left(\mathbf{A}\right)$
- dependent if: $\mathrm{rg}\left(\mathbf{A}\right)$

F37

(**Lecture** 9) S-6 Space spanned by a system of vectors: Generating system

Generating system
The system $\mathsf{Z} = \left[\vec{z}_1; \dots \vec{z}_j;\right]$ **spans** subspace $\mathcal{W}$ if their linear combinations fill $\mathcal{W}$

- $\mathcal{W}$ consists of all linear combinations of $\vec{z}_1, \dots \vec{z}_j$.
- $\mathcal{W}$ is the smallest subspace that contains Z.

$$\mathcal{W} = \mathcal{L}\Big(\left[\vec{z}_1; \dots \vec{z}_j;\right]\Big).$$

Example

- The column space:
$$\mathcal{C}\left(\mathbf{A}\right) = \{\boldsymbol{b} \mid \exists \boldsymbol{x} \text{ such that } \boldsymbol{b} = \mathbf{A}\boldsymbol{x}\} = \mathcal{L}\Big(\text{columns of } \mathbf{A}\Big).$$
- The null space:
$$\mathcal{N}\left(\mathbf{A}\right) = \{\boldsymbol{x} \mid \mathbf{A}\boldsymbol{x} = \mathbf{0}\} = \mathcal{L}\Big(\text{special solutions to } \mathbf{A}\boldsymbol{x} = \mathbf{0}\Big).$$

F38

(**Lecture** 9) S-7 A Basis for a Vector Space

Basis for a subspace $\mathcal{W}$
is a system of vectors $\left[\vec{z}_1; \dots \vec{z}_d;\right]$ such that;

1. span the subspace $\mathcal{W}$
2. are *linearly independent*

examples
$\mathbb{R}^3$:

$\left[\boldsymbol{a}_1; \dots \boldsymbol{a}_n;\right]$ is a basis of $\mathbb{R}^n$ if it is an invertible matrix

All bases for a given subspace $\mathcal{W}$ contain the same *number* of vectors

F39

(**Lecture** 9) S-8 Dimension

$\boxed{\text{All bases for a given subspace } \mathcal{S} \text{ contain the same } \textit{number} \text{ of vectors}}$

The dimension of a space is that number

That number indicates the “size” of the space

F40

(**Lecture** 9) S-**9** Examples: $\mathcal{C}(\mathbf{A})$

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 2 & 3 & 1 \end{bmatrix};$$

- do the columns span $\mathcal{C}(\mathbf{A})$?
- are the columns a basis for $\mathcal{C}(\mathbf{A})$?
- What is the rank of $\mathbf{A}$?
- write down some bases for $\mathcal{C}(\mathbf{A})$

$$\boxed{\operatorname{rg}(\mathbf{A}) = \text{num. pivots} = \text{dimension of } \mathcal{C}(\mathbf{A})}$$

F41

(**Lecture** 9) S-**10** Examples: $\mathcal{N}(\mathbf{A})$

$$\underset{m \times n}{\mathbf{A}} = \begin{bmatrix} 1 & 2 & 3 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 2 & 3 & 1 \end{bmatrix}; \qquad \boldsymbol{v} = \begin{pmatrix} -1 \\ -1 \\ 1 \\ 0 \end{pmatrix}$$

- Is $\boldsymbol{v}$ in $\mathcal{N}(\mathbf{A})$?
- Does $\boldsymbol{v}$ span $\mathcal{N}(\mathbf{A})$?
- write down another vector of $\mathcal{N}(\mathbf{A})$ independent of $\boldsymbol{v}$.
- do $\boldsymbol{v}$ and $\boldsymbol{w}$ span $\mathcal{N}(\mathbf{A})$?
- are $\boldsymbol{v}$ and $\boldsymbol{w}$ a basis for $\mathcal{N}(\mathbf{A})$?
- What is the dimension of $\mathcal{N}(\mathbf{A})$?

$$\boxed{n - \operatorname{rg}(\mathbf{A}) = \text{ num. free variables} = \dim \mathcal{N}(\mathbf{A})}$$

F42

The lecture ends here

Questions of the Lecture 9

(L-9) Question 1. Decide whether or not the following vectors are linearly independent, by solving $c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2 + c_3\boldsymbol{v}_3 + c_4\boldsymbol{v}_4 = \mathbf{0}$:

$$\boldsymbol{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad \boldsymbol{v}_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad \boldsymbol{v}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}, \quad \boldsymbol{v}_4 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}.$$

Decide also if they span $\mathbb{R}^4$, by trying to solve $c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2 + c_3\boldsymbol{v}_3 + c_4\boldsymbol{v}_4 = \begin{pmatrix} 0, & 0, & 0, & 1, \end{pmatrix}$.
(Strang, 2006, exercise 16 from section 2.3.)

(L-9) Question 2. Suppose $\boldsymbol{v}_1 \ldots \boldsymbol{v}_6$ are six vectors in $\mathbb{R}^4$.
(a) Those vectors (do)(do not)(might not) span $\mathbb{R}^4$.
(b) Those vectors (are)(are not)(might be) linearly independent.
(c) If those vectors are the columns of $\mathbf{A}$, then $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$ (has) (does not have) (might not have) a solution.

(d) If those vectors are the columns of $\mathbf{A}$, then $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$ (has) (does not have) (might not have) a sole solution.
(Strang, 2006, exercise 22 from section 2.3.)

(L-9) Question 3. Find $\mathbf{B}$ and $\mathbf{C}$ with the given property or explain why you can't.

(a) The complete solution to $\mathbf{B}\boldsymbol{x} = \begin{pmatrix}1\\2\\4\end{pmatrix}$ is $\boldsymbol{x} = \begin{pmatrix}1\\0\end{pmatrix}$. Find $\mathbf{B}$ or explain why you can't.

(b) The complete solution to $\mathbf{C}\boldsymbol{x} = \begin{pmatrix}5\\1\end{pmatrix}$ is $\boldsymbol{x} = \begin{pmatrix}1\\4\\-2\end{pmatrix}$. Find $\mathbf{C}$ or explain why you can't.

(L-9) Question 4. Show that $\boldsymbol{v}_1,\ \boldsymbol{v}_2,\ \boldsymbol{v}_3$ are independent but $\boldsymbol{v}_1,\ \boldsymbol{v}_2,\ \boldsymbol{v}_3,\ \boldsymbol{v}_4$ are dependent:

$$\boldsymbol{v}_1 = \begin{pmatrix}1\\0\\0\end{pmatrix};\quad \boldsymbol{v}_2 = \begin{pmatrix}1\\1\\0\end{pmatrix};\quad \boldsymbol{v}_3 = \begin{pmatrix}1\\1\\1\end{pmatrix};\quad \boldsymbol{v}_4 = \begin{pmatrix}2\\3\\4\end{pmatrix}.$$

Solve $\mathbf{A}\boldsymbol{c} = \mathbf{0}$ (where the $\boldsymbol{v}$'s go in the columns of $\mathbf{A}$).
(Strang, 2006, exercise 1 from section 2.3.)

(L-9) Question 5. True or false?

If $\mathbf{A}^\intercal = 2\mathbf{A}$, then the rows of $\mathbf{A}$ are linearly dependent.

(L-9) Question 6. Which of the following sets of vectors span $\mathbb{R}^3$?
(a) $(1, 2, 0,)$ and $(0, -1, 1,)$.
(b) $(1, 1, 0,)$, $(0, 1, -2,)$, and $(1, 3, 1,)$.
(c) $(-1, 2, 3,)$, $(2, 1, -1,)$, and $(4, 7, 3,)$.
(d) $(1, 0, 2,)$, $(0, 1, 0,)$, $(-1, 3, 0,)$, and $(1, -4, 1,)$.

(L-9) Question 7. Which of the following systems of vectors are linearly independent? In case of linear dependence, write one of the vectors as a linear combination of the others.
(a) $(-1, 2, 3,)$, $(2, 1, -1,)$, and $(4, 7, 3,)$ in $\mathbb{R}^3$.
(b) $(1, 2, 0,)$ and $(0, -1, 1,)$ in $\mathbb{R}^3$.
(c) $(1, 2,)$, $(2, 3,)$, and $(8, -2,)$ in $\mathbb{R}^2$.
(d) $t^2 + 2t + 1,\quad t^3 - t^2,\quad t^3 + 1,$ and $t^3 + t + 1$ in P_3.

(L-9) Question 8. Suppose the only solution to $\underset{m\times n}{\mathbf{A}}\ \boldsymbol{x} = \mathbf{0}$ (m equations in n unknowns) is $\boldsymbol{x} = \mathbf{0}$. What is the rank and why? The columns of A are linearly ____________.
(Strang, 2006, exercise 8 from section 2.4.)

(L-9) Question 9. **[Important]** If $\mathbf{A}$ has order 4×6, prove that the columns of $\mathbf{A}$ are linearly dependent.

(L-9) Question 10. $\mathbf{A}$ is such that $\mathcal{N}\left(\mathbf{A}\right) = \mathcal{L}\left(\left[\begin{pmatrix}1\\2\\-1\\3\end{pmatrix};\ \begin{pmatrix}0\\1\\1\\4\end{pmatrix};\ \begin{pmatrix}-1\\-1\\3\\1\end{pmatrix};\right]\right)$.

(a) Find a matrix $\mathbf{B}$ such that its column space $\mathcal{C}\left(\mathbf{B}\right) = \mathcal{N}\left(\mathbf{A}\right)$. [Thus, any vector $\boldsymbol{y} \in \mathcal{N}\left(A\right)$ satisfies $\mathbf{B}\boldsymbol{u} = \boldsymbol{y}$ for some $\boldsymbol{u}$.]
(b) Give a different possible answer to (a): another $\mathbf{B}$ with $\mathcal{C}\left(\mathbf{B}\right) = \mathcal{N}\left(\mathbf{A}\right)$.
(c) For some vector $\boldsymbol{b}$, you are told that a particular solution to $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$ is

$$\boldsymbol{x}_p = \begin{pmatrix}1, & 2, & 3, & 4,\end{pmatrix}$$

Now, your classmate Zarkon tells you that a second solution is:

$$\boldsymbol{x}_Z = \begin{pmatrix}1, & 1, & 3, & 0,\end{pmatrix}$$

while your other classmate Hastur tells you "No, Zarkon's solution can't be right, but here's a second solution that is correct:"

$$\boldsymbol{x}_H = \begin{pmatrix}1, & 1, & 3, & 1,\end{pmatrix}$$

Is Zarkon's solution correct, or Hastur's solution, or are both correct? (Hint: what should be true of $\boldsymbol{x}$ - $\boldsymbol{x}_p$ if $\boldsymbol{x}$ is a valid solution?)

MIT Course 18.06 Quiz 1, Spring, 2009

(L-9) QUESTION 11. Consider the matrix

$$\mathsf{A} = \begin{bmatrix} 1 & 2 & -1 & 0 & 0 \\ 1 & 2 & 0 & 2 & 2 \\ 1 & 2 & -1 & 0 & 0 \\ 2 & 4 & 0 & 4 & 4 \end{bmatrix}$$

(a) Find a basis of the column space $\mathcal{C}\left(\mathsf{A}\right)$.
(b) Find a basis of the nullspace $\mathcal{N}\left(\mathsf{A}\right)$.
(c) Find linear conditions on a, b, c, d that guarantee that the system $\mathsf{A}\boldsymbol{x} = (a, b, c, d,)$ has a solution.
(d) Find the complete solution for the system $\mathsf{A}\boldsymbol{x} = \begin{pmatrix}0\\1\\0\\2\end{pmatrix}$.

MIT Course 18.06 Quiz 1, March 5, 2007

(L-9) QUESTION 12. Si a una matriz A se le "añade" una nueva columna extra $\boldsymbol{b}$, entonces el espacio columna se vuelve más grande, a no ser que ____________. Proporcione un ejemplo en el que espacio columna se haga más grande, y uno en el que no. ¿Por qué $\mathsf{A}\boldsymbol{x} = \boldsymbol{b}$ es resoluble cuando el espacio columna no crece al añadir $\boldsymbol{b}$?

(L-9) QUESTION 13. If the 9 by 12 system $\mathsf{A}\boldsymbol{x} = \boldsymbol{b}$ is solvable for every $\boldsymbol{b}$, then $\mathcal{C}\left(\mathsf{A}\right) =$ ____
(Strang, 2006, exercise 30 from section 2.1.)

(L-9) QUESTION 14. **[Importante]**[1] Suponga que el sistema $\left[\boldsymbol{v}_1; \; \dots \; \boldsymbol{v}_n\right]$ de vectores de $\mathbb{R}^m$ genera el subespacio $\mathcal{V}$, y suponga que $\boldsymbol{v}_n$ es una combinación lineal de los vectores $\boldsymbol{v}_1, \dots \boldsymbol{v}_{n-1}$. Demuestre que el sistema $\left[\boldsymbol{v}_1; \; \dots \; \boldsymbol{v}_{n-1}\right]$ también genera el subespacio $\mathcal{V}$.

(L-9) QUESTION 15.
(a) Find the general (complete) solution to this equation

$$\begin{bmatrix} 1 & 1 & 2 \\ 1 & 1 & 2 \\ 2 & 2 & 2 \end{bmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 4 \end{pmatrix}$$

(b) Find a basis for the column of the 3 by 9 block matrix $\left[\mathsf{A}; \quad 2\mathsf{A}; \quad \mathsf{A}^2;\right]$.

MIT Course 18.06 Final, May 18, 1998

(L-9) QUESTION 16. ¿Cuáles de los siguientes vectores generan el espacio de polinomios de, a lo sumo, grado 4; es decir, el conjunto de polinomios $P_3 = \{at^3 + bt^2 + ct + d\}$?
(a) $t + 1, \;\; t^2 - t, \;$ y $\; t^3$.
(b) $t^3 + t \;$ y $\; t^2 + 1$.
(c) $t^2 + t + 1, \;\; t + 1, \;\; 1, \;$ y $\; t^3$.
(d) $t^3 + t^2, \;\; t^2 - t, \;\; 2t + 4, \;$ y $\; t^3 + 2t^2 + t + 4$.

(L-9) QUESTION 17. Considere los vectores $\boldsymbol{u}_1 = (1, 0, 1,)$ y $\boldsymbol{u}_2 = (1, -1, 1,)$.
(a) **Demuestre** que $\boldsymbol{u}_1$ y $\boldsymbol{u}_2$ son linealmente independientes.

[1] *pista:* Piense si el espacio $\mathcal{V}$ se puede expresar como el espacio columna de una matriz V cuyas columnas son los vectores $\boldsymbol{v}_1, \dots, \boldsymbol{v}_n$. Una vez expresado de esa manera, recuerde que las operaciones entre columnas no alteran el espacio columna de la matriz. Por último, transforme V de manera que transforme una de las columnas en un vector de ceros.

(b) ¿Pertenece $\boldsymbol{v} = (2, 1, 2,)$ al espacio generado por $\{\boldsymbol{u}_1, \boldsymbol{u}_2\}$? Explique las razones de su respuesta.
(c) Encuentre una base de $\mathbb{R}^3$ que contenga a $\boldsymbol{u}_1$ y a $\boldsymbol{u}_2$. Explique su respuesta.

(L-9) Question 18.

(a) ¿Son linealmente independientes los siguientes vectores? Explique su respuesta.

$$\boldsymbol{v}_1 = \begin{pmatrix} -2 \\ -1 \\ 3 \\ 4 \end{pmatrix}; \qquad \boldsymbol{v}_2 = \begin{pmatrix} -8 \\ 2 \\ -2 \\ 1 \end{pmatrix}$$

(b) ¿Son los siguientes vectores una base de $\mathbb{R}^4$? Explique su respuesta.

$$\boldsymbol{v}_1 = \begin{pmatrix} -2 \\ -1 \\ 3 \\ 4 \end{pmatrix}; \qquad \boldsymbol{v}_2 = \begin{pmatrix} 8 \\ 2 \\ 2 \\ 1 \end{pmatrix}; \qquad \boldsymbol{v}_3 = \begin{pmatrix} 10 \\ 1 \\ 1 \\ 6 \end{pmatrix}; \qquad \boldsymbol{v}_4 = \begin{pmatrix} -2 \\ -1 \\ 3 \\ 4 \end{pmatrix}$$

(c) ¿Son los siguientes vectores una base del subespacio descrito por el plano tridimensional $x_1 + 2x_2 + 3x_3 + 6x_4 = 0$? Explique su respuesta.

$$\boldsymbol{v}_1 = \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix}; \qquad \boldsymbol{v}_2 = \begin{pmatrix} -1 \\ -1 \\ 1 \\ 0 \end{pmatrix}; \qquad \boldsymbol{v}_3 = \begin{pmatrix} -4 \\ -2 \\ 2 \\ 1 \end{pmatrix}$$

(d) Encuentre el valor de q para el que los siguientes vectores no generan $\mathbb{R}^3$.

$$\boldsymbol{v}_1 = \begin{pmatrix} 1 \\ 4 \\ 6 \end{pmatrix}; \qquad \boldsymbol{v}_2 = \begin{pmatrix} 0 \\ 2 \\ 2 \end{pmatrix}; \qquad \boldsymbol{v}_3 = \begin{pmatrix} -1 \\ 12 \\ 10 \end{pmatrix}; \qquad \boldsymbol{v}_4 = \begin{pmatrix} q \\ 3 \\ 1 \end{pmatrix}$$

(L-9) Question 19. Suponga que tiene 4 vectores columna $\boldsymbol{u}$, $\boldsymbol{v}$, $\boldsymbol{w}$ y $\boldsymbol{z}$ en el espacio tridimensional $\mathbb{R}^3$.

(a) Dé un ejemplo donde el espacio columna de $\mathbf{A}$ contenga $\boldsymbol{u}$, $\boldsymbol{v}$ y $\boldsymbol{w}$, pero no a $\boldsymbol{z}$. (escriba unos vectores $\boldsymbol{u}$, $\boldsymbol{v}$, $\boldsymbol{w}$ y $\boldsymbol{z}$; y una matriz $\mathbf{A}$ que cumplan lo anterior).
(b) ¿Cuáles son las dimensiones del espacio columna y del espacio nulo de su matriz ejemplo $\mathbf{A}$ del apartado anterior?

End of Questions of the Lecture 9

LECTURE 10: The Four Fundamental Subspaces of a matrix $\mathbf{A}$

$\boxed{\textit{Lecture 10}}$

(**Lecture** 10) S-1 Highlights of Lesson 10

Highlights of *Lesson 10*

- The Four Fundamental Subspaces of a matrix $\mathbf{A}$
  - Column space $\mathcal{C}(\mathbf{A})$
  - Nullspace $\mathcal{N}(\mathbf{A})$
  - Row space $\mathcal{C}(\mathbf{A}^\mathsf{T})$
  - Left nullspace $\mathcal{N}(\mathbf{A}^\mathsf{T})$

F43

(**Lecture** 10) S-2 The Four Fundamental Subspaces of a matrix $\mathbf{A}$

Where are those subspaces if $\underset{m\times n}{\mathbf{A}}$?

- Column space $\mathcal{C}(\mathbf{A})$
- Nullspace $\mathcal{N}(\mathbf{A})$
- Row space

  Linear combinations of the rows

  Linear combinations of the columns of $\mathbf{A}^\mathsf{T} = \mathcal{C}(\mathbf{A}^\mathsf{T})$
- Left nullspace of $\mathbf{A}$, $\quad \mathcal{N}(\mathbf{A}^\mathsf{T})$

F44

(**Lecture** 10) S-3 Bases for the 4 subspaces: row space

$$\left[\begin{array}{c}\mathbf{A}\\ \hline \mathbf{I}\end{array}\right] = \left[\begin{array}{cccc}1&2&3&1\\1&1&2&1\\1&2&3&1\\ \hline 1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{array}\right] \xrightarrow[{[(-1)\mathbf{1}+\mathbf{4}]}]{\substack{\tau\\ [(-2)\mathbf{1}+\mathbf{2}]\\ [(-3)\mathbf{1}+\mathbf{3}]}} \left[\begin{array}{cccc}1&0&0&0\\1&-1&-1&0\\1&0&0&0\\ \hline 1&-2&-3&-1\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{array}\right] \xrightarrow[{[(1)\mathbf{2}+\mathbf{3}]}]{\substack{\tau\\ [(-1)\mathbf{2}]\\ [(-1)\mathbf{2}+\mathbf{1}]}} \left[\begin{array}{cccc}1&0&0&0\\0&1&0&0\\1&0&0&0\\ \hline -1&2&-1&-1\\1&-1&-1&0\\0&0&1&0\\0&0&0&1\end{array}\right] = \left[\begin{array}{c}\mathbf{R}\\ \hline \mathbf{E}\end{array}\right]$$

column operations preserve $\mathcal{C}(\mathbf{A})$ (but not the row space)

$$\mathcal{C}(\mathbf{A}^\mathsf{T}) \neq \mathcal{C}(\mathbf{L}^\mathsf{T}) \neq \mathcal{C}(\mathbf{R}^\mathsf{T}); \quad (1,\ \ 2,\ \ 3,\ \ 1,) \in \mathcal{C}(\mathbf{A}^\mathsf{T}) \text{ but } \notin \mathcal{C}(\mathbf{R}^\mathsf{T})$$

What's the dimension of the row space $\mathcal{C}(\mathbf{A}^\mathsf{T})$?
What's a basis for the row space of $\mathbf{A}$?
A basis for the column space of $\mathbf{A}$?

F45

(**Lecture** 10) **S-4** Left null space: why that name?

$\mathcal{N}(\mathbf{A}^\mathsf{T})$

$$(\mathbf{A}^\mathsf{T})\boldsymbol{y} = \mathbf{0}$$

$$\begin{bmatrix} & \mathbf{A}^\mathsf{T} & \end{bmatrix}\begin{pmatrix} y_1 \\ \vdots \\ y_m \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$$

so. . .

$$\boldsymbol{y}\mathbf{A} = \mathbf{0}$$

$$(y_1, \quad \dots \quad y_m,)\begin{bmatrix} & \mathbf{A} & \end{bmatrix} = (0, \quad \dots \quad 0,)$$

F46

(**Lecture** 10) **S-5** Column elimination preserves the left null space

Let $\underset{n\times n}{\mathbf{E}} = \mathbf{I}_{\tau_1\cdots\tau_k}$ *be invertible* then

- *If* $\boldsymbol{x} \in \mathcal{N}(\mathbf{A}^\mathsf{T})$
$$\boldsymbol{x}\mathbf{A} = \mathbf{0} \quad \text{and} \quad \boldsymbol{x}\mathbf{AE} = \mathbf{0E} = \mathbf{0} \quad \Rightarrow \boldsymbol{x} \in \mathcal{N}((\mathbf{AE})^\mathsf{T});$$
- *If* $\boldsymbol{x} \in \mathcal{N}((\mathbf{AE})^\mathsf{T})$
$$\boldsymbol{x}\mathbf{AE} = \mathbf{0} \quad \text{and} \quad \boldsymbol{x}\mathbf{A} = \mathbf{0E}^{-1} = \mathbf{0} \quad \Rightarrow \boldsymbol{x} \in \mathcal{N}(\mathbf{A}^\mathsf{T}).$$

Therefore,

$$\mathcal{N}(\mathbf{A}^\mathsf{T}) = \mathcal{N}((\mathbf{AE})^\mathsf{T}) = \mathcal{N}\left((\mathbf{A}_{\tau_1\cdots\tau_k})^\mathsf{T}\right).$$

F47

(**Lecture** 10) **S-6** Finding a basis of $\mathcal{N}(\mathbf{A}^\mathsf{T})$ by column reduction

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 2 & 3 & 1 \end{bmatrix} \xrightarrow[\substack{[(-2)\mathbf{1}+\mathbf{2}] \\ [(-3)\mathbf{1}+\mathbf{3}] \\ [(-1)\mathbf{1}+\mathbf{4}]}]{\boldsymbol{\tau}} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & \text{-}1 & \text{-}1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \xrightarrow[\substack{[(-1)\mathbf{2}] \\ [(-1)\mathbf{2}+\mathbf{1}] \\ [(1)\mathbf{2}+\mathbf{3}]}]{\boldsymbol{\tau}} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} = \mathbf{R}$$

Basis for $\mathcal{N}(\mathbf{A}^\mathsf{T})$?

$$[(-1, \quad 0, \quad 1,);]$$

F48

(**Lecture** 10) **S-7** Finding a basis of $\mathcal{N}(\mathbf{A}^\mathsf{T})$ by column reduction

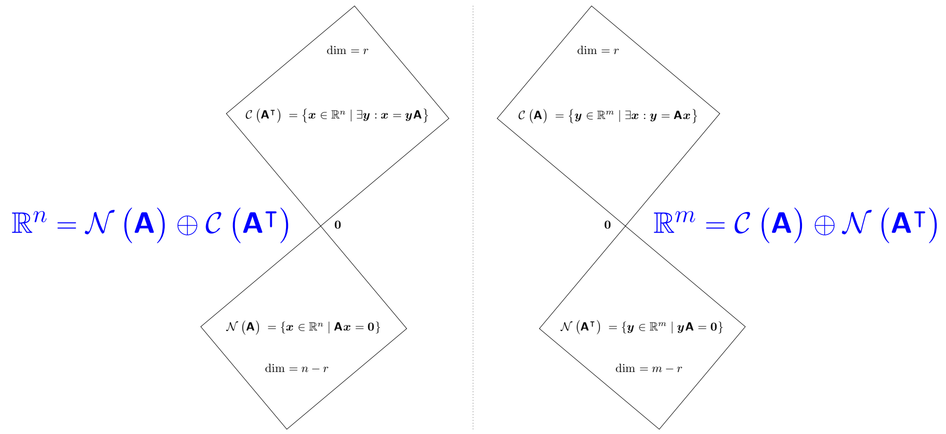
$$\begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 1 \\ 0 & 0 & 1 \\ a & b & c \\ d & e & f \end{bmatrix} \xrightarrow[{[(1)\mathbf{1}+\mathbf{3}]}]{\boldsymbol{\tau}} \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \\ a & b & a+c \\ d & e & d+f \end{bmatrix} \xrightarrow[{[(1)\mathbf{2}+\mathbf{1}]}]{\boldsymbol{\tau}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ a+b & b & a+c \\ d+e & e & d+f \end{bmatrix}$$

Basis for $\mathcal{N}(\mathbf{A}^\mathsf{T})$?

F49

$$\text{Base de } \mathcal{N}\left(\mathbf{A}^{\intercal}\right) = \left[\begin{pmatrix} -a-b \\ -b \\ -a-c \\ 1 \\ 0 \end{pmatrix}; \begin{pmatrix} -d-e \\ -e \\ -d-f \\ 0 \\ 1 \end{pmatrix}; \right]$$

(**Lecture** 10) S-8 **The Four Fundamental Subspaces**



$\underset{m\times n}{\mathbf{A}}$ dimensions?

- $\dim\left(\mathcal{C}\left(\mathbf{A}\right)\right) =$ $\qquad\qquad\qquad = \dim\left(\mathcal{C}\left(\mathbf{A}^{\intercal}\right)\right)$
- $\dim\left(\mathcal{N}\left(\mathbf{A}\right)\right) =$
- $\dim\left(\mathcal{N}\left(\mathbf{A}^{\intercal}\right)\right) =$

F50

The lecture ends here

Questions of the Lecture 10

(L-10) Question 1. Find the dimension and construct a basis for the four subspaces associated with each of the matrices

(a) $\mathbf{A} = \begin{bmatrix} 0 & 1 & 4 & 0 \\ 0 & 2 & 8 & 0 \end{bmatrix}$

(b) What is the sum $\dim \mathcal{C}\left(\mathbf{A}\right) + \dim \mathcal{N}\left(\mathbf{A}^{\intercal}\right)$? and $\dim \mathcal{C}\left(\mathbf{A}^{\intercal}\right) + \dim \mathcal{N}\left(\mathbf{A}\right)$?

(c) $\mathbf{U} = \begin{bmatrix} 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

(d) What is the sum $\dim \mathcal{C}\left(\mathbf{U}\right) + \dim \mathcal{N}\left(\mathbf{U}^{\intercal}\right)$? and $\dim \mathcal{C}\left(\mathbf{U}^{\intercal}\right) + \dim \mathcal{N}\left(\mathbf{U}\right)$?

(Strang, 2006, exercise 2 from section 2.4.)

(L-10) Question 2. Describe the four subspaces in three-dimensional space associated with

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

(Strang, 2006, exercise 4 from section 2.4.)

(L-10) Question 3.

(a) If a 7 by 9 matrix has rank 5, what are the dimensions of the four subspaces? What is the sum of all four dimensions?
(b) If a 3 by 4 matrix has rank 3, what are its column space $\mathcal{C}\left(\mathbf{A}\right)$ and left nullspace $\mathcal{N}\left(\mathbf{A}^{\mathsf{T}}\right)$?

(Strang, 2006, exercise 20 from section 2.4.)

(L-10) Question 4. If $\mathbf{A}$ has the same four fundamental subspaces as $\mathbf{B}$, does $\mathbf{A} = c\mathbf{B}$?
(Strang, 2006, exercise 19 from section 2.4.)

(L-10) Question 5. Find the dimension and a basis for the four fundamental subspaces for

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{bmatrix}; \qquad \mathbf{L} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} = \mathbf{AE}; \quad \text{where} \quad \mathbf{E} = \begin{bmatrix} 1 & -2 & 2 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Basado en (Strang, 2006, exercise 3 from section 2.4.)

(L-10) Question 6. Find the dimensions of these vector spaces:
(a) The space of all vectors in $\mathbb{R}^4$ whose components add to zero.
(b) The nullspace of the 4 by 4 identity matrix.
(c) The space of all 4 by 4 matrices

(Strang, 2006, exercise 32 from section 2.3.)

(L-10) Question 7. Without multiplying matrices, find bases for the row and column spaces of $\mathbf{A}$:

$$\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 4 & 5 \\ 2 & 7 \end{bmatrix} \begin{bmatrix} 3 & 0 & 3 \\ 1 & 1 & 2 \end{bmatrix}.$$

How do you know from these shapes that $\mathbf{A}$ is not invertible?
(Strang, 2006, exercise 36 from section 2.4.)

(L-10) Question 8. Which of the following (if any) are subspaces? For any that are not a subspace, give an example of how they violate a property of subspaces.
(a) Given 3×5 matrix $\mathbf{A}$ with full row rank, the set of all solutions to

$$\mathbf{A}\boldsymbol{x} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

(b) All vectors $\boldsymbol{x}$ with $\left\langle \vec{x} \middle| \vec{y} \right\rangle = \mathbf{0}$ and $\left\langle \vec{x} \middle| \vec{z} \right\rangle = \mathbf{0}$ for some given vectors $\boldsymbol{y}$ and $\boldsymbol{z}$.
(c) All 3×5 matrices with $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$. in their column space.
(d) All 5×3 matrices with $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ in their nullspace.

MIT Course 18.06 Quiz 1, Spring, 2009

(L-10) Question 9. ¿Cuál es el espacio columna $\mathcal{C}\left(\mathbf{A}\right)$ y el espacio fila $\mathcal{C}\left(\mathbf{A}^{\mathsf{T}}\right)$ de la matriz

$$\mathbf{A} = \begin{bmatrix} 1 & 3 & 5 & -2 \\ 2 & -1 & 3 & -4 \\ -1 & 4 & 2 & 2 \end{bmatrix}$$

MIT Course 18.06 Final. May 18, 1998

(L-10) Question 10. If $\underset{5\times 4}{\mathbf{A}}$ is a matrix with linearly independent columns, find each of these explicitily:

(a) The nulls space of $\mathbf{A}$.
(b) The dimension of the left null space $\mathcal{N}\left(\mathbf{A}^{\intercal}\right)$.
(c) One particular solution $\boldsymbol{x}_p$ to the system $\mathbf{A}\boldsymbol{x} = \mathbf{A}_{|2}$.
(d) The general (complete) solution to $\mathbf{A}\boldsymbol{x} = \mathbf{A}_{|2}$.
(e) The reduced echelon form $\mathbf{R}$ of $\mathbf{A}$.

(L-10) Question 11. Verdadero o falso

(a) Si una matriz es cuadrada $(m = n)$, entonces el espacio columna es igual al espacio fila.
(b) La matriz $\mathbf{A}$ y la matriz $(-\mathbf{A})$ comparten los mismos cuatro sub-espacios fundamentales.
(c) Si $\mathbf{A}$ y $\mathbf{B}$ comparten los mismos cuatro sub-espacios fundamentales, entonces $\mathbf{A}$ es un múltiplo de $\mathbf{B}$.
(d) Indique si la siguiente aseveración es verdadera o falsa. Si es verdadera explique el motivo, si es falsa encuentre un contraejemplo: *"Un sistema con n ecuaciones y n incógnitas es resoluble cuando las columnas de la matriz de coeficientes son independientes."*

(L-10) Question 12. Se conoce la siguiente información sobre $\mathbf{A}$:

$$\mathbf{A}\boldsymbol{v} = \mathbf{A}\begin{pmatrix}1\\-2\\3\\1\end{pmatrix} = \begin{pmatrix}-6\\3\end{pmatrix}; \qquad \text{y que} \qquad \mathbf{A}\boldsymbol{w} = \mathbf{A}\begin{pmatrix}3\\-1\\1\\2\end{pmatrix} = \begin{pmatrix}-18\\9\end{pmatrix}.$$

De hecho, $\mathbf{A}\boldsymbol{x}$ es siempre algún múltiplo del vector $\begin{pmatrix}-2, & 1,\end{pmatrix}$ sea cual sea el vector $\boldsymbol{x} \in \mathbb{R}^4$.

(a) ¿Cuál es el orden y el rango de $\mathbf{A}$?
(b) ¿Cuál es la dimensión del espacio nulo $\mathcal{N}\left(\mathbf{A}\right)$?
(c) ¿Cuál es la dimensión del espacio fila $\mathcal{C}\left(\mathbf{A}^{\intercal}\right)$?
(d) ¿Cuál es la dimensión del espacio nulo por la izquierda $\mathcal{N}\left(\mathbf{A}^{\intercal}\right)$?
(e) Encuentre una solución $\boldsymbol{x}$ no nula al sistema $\mathbf{A}\boldsymbol{x} = \begin{pmatrix}0\\0\end{pmatrix}$.

(L-10) Question 13. Consider the matrix $\mathbf{A}$ with its column reduced echelon form $\mathbf{R}$ computed by gaussian elimination without permutations:

$$\left[\frac{\mathbf{A}}{\mathbf{I}}\right] = \left[\begin{array}{rrrrr}1 & -1 & 5 & 1 & 0\\ 2 & 1 & 4 & 2 & 1\\ 3 & 0 & 9 & 3 & 1\\ -1 & -1 & -1 & -1 & 2\\ \hline 1 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 1\end{array}\right] \to \left[\begin{array}{rrrrr}1 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 0\\ 1 & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 1\\ \hline 3/8 & 2/8 & -3 & -1 & -1/8\\ -5/8 & 2/8 & 2 & 0 & -1/8\\ 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1 & 0\\ -1/8 & 2/8 & 0 & 0 & 3/8\end{array}\right] = \left[\frac{\mathbf{R}}{\mathbf{E}}\right]$$

(a) What is the rank of $\mathbf{A}$? What are the dimensions of the column space $\mathcal{C}\left(\mathbf{A}\right)$, the row space $\mathcal{C}\left(\mathbf{A}^{\intercal}\right)$ and the nullspace $\mathcal{N}\left(\mathbf{A}\right)$?
(b) Find a basis of the row space $\mathcal{C}\left(\mathbf{A}^{\intercal}\right)$.
(c) Find a basis for the column space $\mathcal{C}\left(\mathbf{A}\right)$.
(d) Find a basis for the nullspace $\mathcal{N}\left(\mathbf{A}\right)$.
(e) Write down $\mathbf{A}_{|3}$ as a linear combination of $\mathbf{A}_{|1}$, $\mathbf{A}_{|2}$, $\mathbf{A}_{|4}$ and $\mathbf{A}_{|5}$.

(L-10) Question 14. Construct a matrix whose nullspace consists of all combinations of $(2, 2, 1, 0,)$ and $(3, 1, 0, 1,)$. (Strang, 2006, exercise 60 from section 2.2.)

(L-10) Question 15. Construct a matrix whose nullspace consists of all multiples of $(4, 3, 2, 1)^{\intercal}$
(Strang, 2006, exercise 61 from section 2.2.)

(L-10) Question 16.

(a) Suponga que el producto de $\mathbf{A}$ y $\mathbf{B}$ es la matriz nula: $\mathbf{AB} = \mathbf{0}$. Entonces el espacio (I)________________ de la matriz $\mathbf{A}$ contiene el espacio (II)________________ de la matriz $\mathbf{B}$. También el espacio (III)________________ de la matriz $\mathbf{B}$ contiene el espacio (IV)________________ de la matriz $\mathbf{A}$. (incluya los nombres de los cuatro espacios fundamentales en los lugares apropiados)

(I)________________, (II)________________, (III)________________, (IV)________________

(b) Suponga que la matriz $\mathbf{A}$ es de dimensiones 5 por 7 con rango r, y $\mathbf{B}$ es de dimensiones 7 por 9 de rango s. ¿Cuáles son las dimensiones de los espacios (I) y (II)? Del hecho de que el espacio (I) contiene el espacio (II), ¿qué sabe acerca de $r + s$?

(L-10) QUESTION 17. By performing column eliminations (and possibly permutations) on the following 4×8 matrix $\mathbf{A}$

$$\left[\frac{\mathbf{A}}{\mathbf{I}}\right] = \left[\begin{array}{rrrrrrrr}
1 & 2 & -2 & 1 & -5 & 0 & 2 & -3\\
-1 & -2 & 1 & -2 & 3 & 0 & -2 & 0\\
1 & 2 & -2 & 1 & -5 & 1 & 1 & 1\\
0 & 0 & 2 & 2 & 4 & 0 & 0 & 2\\
\hline
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{array}\right] \to \left[\begin{array}{rrrrrrrr}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1\\
\hline
2/4 & -2 & -2/4 & -3 & 1 & 0 & -2 & 3/4\\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0\\
2/4 & 0 & 2/4 & -1 & -2 & 0 & 0 & 3/4\\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0\\
1 & 0 & 2 & 0 & 0 & 1 & 1 & 1\\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0\\
-2/4 & 0 & -2/4 & 0 & 0 & 0 & 0 & -1/4
\end{array}\right] = \left[\frac{\mathbf{R}}{\mathbf{E}}\right]$$

(a) What is the rank of $\mathbf{A}$?
(b) What are the dimensions of the 4 fundamental subspaces?
(c) How many solutions does $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$ have? Does it depend on $\boldsymbol{b}$? Justify.
(d) Are the rows of $\mathbf{A}$ linearly independent? Why?
(e) Give a basis of $\mathcal{N}\left(\mathbf{A}\right)$.
(f) Give a basis of $\mathcal{N}\left(\mathbf{A}^{\intercal}\right)$.
(g) Give, if possible, matrix $\left[\mathbf{A}_{|1};\ \mathbf{A}_{|3};\ \mathbf{A}_{|6};\ \mathbf{A}_{|7};\right]^{-1}$
(h) Give, if possible, matrix $\left[\mathbf{A}_{|1};\ \mathbf{A}_{|3};\ \mathbf{A}_{|6};\ \mathbf{A}_{|8};\right]^{-1}$

Based on *MIT Course 18.06 Quiz 1, October 4, 2004*

(L-10) QUESTION 18. Consider the 5 by 3 matrix $\mathbf{R}$ (in its column reduced echelon form) with three pivots ($r = 3$).
(a) What is its null space $\mathcal{N}\left(\mathbf{R}\right)$?
(b) Consider the 10 by 3 block matrix; $\mathbf{B} = \begin{bmatrix}\mathbf{R}\\ 2\mathbf{R}\end{bmatrix}$. What is its column reduced echelon form? What is its rank?
(c) Consider the 10 by 6 block matrix; $\mathbf{C} = \begin{bmatrix}\mathbf{R} & \mathbf{R}\\ \mathbf{R} & \mathbf{0}\end{bmatrix}$. What is its column reduced echelon form?
(d) What is the rank of $\mathbf{C}$?
(e) What is the dimension of the null space of $\mathbf{C}^{\intercal}$; $\dim \mathcal{N}\left(\mathbf{C}^{\intercal}\right)$?

Based on MIT Course 18.06 Quiz 1, Fall 1993

(L-10) QUESTION 19. Consider the linear system $\mathbf{A}\boldsymbol{x} = \begin{pmatrix}2\\4\\2\end{pmatrix}$

$$\text{with solution } \left\{ \boldsymbol{x} \in \mathbb{R}^3 \;\middle|\; \exists\, c, d \in \mathbb{R} \text{ such that } \boldsymbol{x} = \begin{pmatrix}2\\0\\0\end{pmatrix} + c\begin{pmatrix}1\\1\\0\end{pmatrix} + d\begin{pmatrix}0\\0\\1\end{pmatrix} \right\}.$$

(a) (1^{pts}) Find the dimension of the row space of $\mathbf{A}$. Explain your answer.

(b) (1^{pts}) Construct the matrix $\mathbf{A}$. Explain your answer.
(c) (0.5^{pts}) For which right hand side vectors $\boldsymbol{b}$ the system $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$ is solvable?

(L-10) QUESTION 20. True or false (give a good reason)?

(a) If the columns of a matrix are dependent, so are the rows.
(b) The column space of a 2 by 2 matrix is the same as its row space.
(c) The column space of a 2 by 2 matrix has the same dimension as its row space.
(d) The columns of a matrix are a basis for the column space.

(Strang, 2006, exercise 28 from section 2.3.)

(L-10) QUESTION 21. Let $\mathbf{A}$ be any matrix and $\mathbf{R}$ its **row reduced** echelon form. Answer True or False to the statements below and briefly explain. (Note, if there are any counterexamples to a statement below you must choose false for that statement.)

(a) If $\boldsymbol{x}$ is a solution to $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$ then $\boldsymbol{x}$ must be a solution to $\mathbf{R}\boldsymbol{x} = \boldsymbol{b}$.
(b) If $\boldsymbol{x}$ is a solution to $\mathbf{A}\boldsymbol{x} = \mathbf{0}$ then $\boldsymbol{x}$ must be a solution to $\mathbf{R}\boldsymbol{x} = \mathbf{0}$.
(c) What would be your anwsers if $\mathbf{R}$ is the **column** reduced form of $\mathbf{A}$?

basado en MIT Course 18.06 Quiz 1, Fall 2008

End of Questions of the Lecture 10

OPTIONAL LECTURE I: Review

$\boxed{\textit{Lecture}}$

(**Lecture** 10) S-1 Highlights of Lesson

Highlights of *Lesson*

- Bases of new vector spaces
- Rank one matrices
- Free variables

F51

(**Lecture** 10) S-2 A new vector space

$\mathbb{R}^{3\times 3}$: All matrices of order 3×3! $\quad \mathbf{A}+\mathbf{B}; \quad c\mathbf{A}; \quad \underset{3\times 3}{\mathbf{0}}$

subspaces of $\mathbb{R}^{3\times 3}$

- $\mathcal{U}$: Upper triangular matrices
- $\mathcal{S}$: Symmetric matrices
- $\mathcal{U}\cap\mathcal{S}$: The intersection (the vectors that are in both $\mathcal{S}$ and $\mathcal{U}$):

What are the dimensions of these subspaces?
Is $\mathcal{U}\cup\mathcal{S}$ a subspace?
Let $\mathcal{U}+\mathcal{S}$ be the set of all sums of vectors in $\mathcal{U}$ plus vectors in $\mathcal{S}$; then $\mathcal{U}+\mathcal{S}=\quad$?

F52

(**Lecture** 10) S-3 Rank one matrices

$$\mathbf{A}=\begin{bmatrix}1 & 4 & 5\\ & & \end{bmatrix}$$

- tell me a basis for the row space:
- tell me a basis for the column space

What's the dimension of $\mathcal{C}\left(\mathbf{A}\right)$, $\mathcal{C}\left(\mathbf{A}^{\intercal}\right)$ and rg $\left(\mathbf{A}\right)$?

$$\mathbf{A}=\begin{bmatrix}1 & 4 & 5\\ & & \end{bmatrix}=\begin{bmatrix}1\\ \ \end{bmatrix}\begin{bmatrix}1 & 4 & 5\end{bmatrix}$$

Every rank one matrix has the form: a column times a row.

$$\mathbf{A}=\begin{bmatrix}\mathbf{A}_{|1}\end{bmatrix}\begin{bmatrix}{}_{1|}\mathbf{A}\end{bmatrix}^{\intercal}=\text{column matrix times row matrix}$$

F53

- Another way to denote the column matrix is $\mathbf{A}_{|(1,)}$
- Another way to denote the row matrix ${}_{(1,)|}\mathbf{A}$

Note that the argument in the picking operator is not a number, but a vector of $\mathbb{R}^1$, i.e. a list with a single number (see Appendix 3.B of Lesson 3 of the book)..

Hence $\mathbf{A} = \left[\mathbf{A}_{|1}\right]\left[{}_{1|}\mathbf{A}\right]^{\mathsf{T}} = \left(\mathbf{A}_{|(1,)}\right)\left({}_{(1,)|}\mathbf{A}\right) = \begin{bmatrix}1\\2\end{bmatrix}\begin{bmatrix}1 & 4 & 5\end{bmatrix}.$

(**Lecture** 10) S-4 **Rank one matrices**

Think about the following subset of $\mathbb{R}^4$:

$$\mathcal{S} = \left\{ \boldsymbol{v} = \begin{pmatrix} v_1\\ v_2\\ v_3\\ v_4 \end{pmatrix} \in \mathbb{R}^4 \;\middle|\; v_1 + v_2 + v_3 + v_4 = 0 \right\}$$

Is $\mathcal{S}$ a subspace?
What's the dimension? can you tell me a basis?
$\mathcal{S}$ is the null space of a matrix $\mathbf{A}$ $\quad (\mathbf{A}\boldsymbol{v} = 0)\ldots$ Which matrix?

F54

(**Lecture** 10) S-5 **Rank one matrices**

$$\underset{1\times 4}{\mathbf{A}} = \begin{bmatrix}1 & 1 & 1 & 1\end{bmatrix}; \qquad \operatorname{rg}(\mathbf{A}) = \qquad\qquad \mathcal{S} = \mathcal{N}(\mathbf{A})$$

- $\dim \mathcal{S} = \dim \mathcal{N}(\mathbf{A}) =$
- basis of $\mathcal{S} = \mathcal{N}(\mathbf{A})$?

$$\left[\begin{pmatrix}1\\0\\0\end{pmatrix}; \quad \begin{pmatrix}0\\1\\0\end{pmatrix}; \quad \begin{pmatrix}0\\0\\1\end{pmatrix};\right]$$

- $\dim \mathcal{C}\left(\mathbf{A}^{\mathsf{T}}\right) =$
- $\dim \mathcal{N}\left(\mathbf{A}^{\mathsf{T}}\right) =$ $\qquad$ basis $\mathcal{N}\left(\mathbf{A}^{\mathsf{T}}\right)$?
- $\dim \mathcal{C}(\mathbf{A}) =$

F55

(**Lecture** 10) S-6 **A problem from Microeconomics**

Solve Y in terms of X to get PPF

$$\begin{cases} X & & = 4L_x \\ & Y & = 3L_y \\ & & L_x + L_y = 80 \end{cases} \rightarrow \begin{cases} X & & -4L_x & & = 0 \\ & Y & & -3L_y & = 0 \\ & & L_x + & L_y & = 80 \end{cases}$$

(*"in terms of" X means X free*)

$$\left[\begin{array}{cccc|c} 1 & 0 & -4 & 0 & 0 \\ 0 & 1 & 0 & -3 & 0 \\ 0 & 0 & 1 & 1 & -80 \\ \hline 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow[{[(3)\mathbf{2}+\mathbf{4}]}]{\tau \atop {[(4)\mathbf{1}+\mathbf{3}]}} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & -80 \\ \hline 1 & 0 & 4 & 0 & 0 \\ 0 & 1 & 0 & 3 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow[{[(80)\mathbf{3}+\mathbf{4}]}]{\tau \atop {[(-1)\mathbf{3}+\mathbf{4}]}} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ \hline 1 & 0 & 4 & -4 & 320 \\ 0 & 1 & 0 & 3 & 0 \\ 0 & 0 & 1 & -1 & 80 \\ 0 & 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\begin{pmatrix} X \\ Y \\ L_x \\ L_y \end{pmatrix} = \begin{pmatrix} 320 \\ 0 \\ 80 \\ 0 \end{pmatrix} + a \begin{pmatrix} -4 \\ 3 \\ -1 \\ 1 \end{pmatrix} \Rightarrow a = L_y \quad \Rightarrow \quad \begin{pmatrix} X \\ Y \\ L_x \\ L_y \end{pmatrix} = \begin{pmatrix} 320 - 4L_y \\ 3L_y \\ 80 - L_y \\ L_y \end{pmatrix}$$

"in terms of" L_y

F56

(**Lecture** 10) S-7 **Free variable**

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ \hline 1 & 0 & 4 & -4 & 320 \\ 0 & 1 & 0 & 3 & 0 \\ 0 & 0 & 1 & -1 & 80 \\ 0 & 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow[{[(-320)\mathbf{4}+\mathbf{5}]}]{\tau \atop {[(\frac{-1}{4})\mathbf{4}]}} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ \hline 1 & 0 & 4 & 1 & 0 \\ 0 & 1 & 0 & -3/4 & 240 \\ 0 & 0 & 1 & 1/4 & 0 \\ 0 & 0 & 0 & -1/4 & 80 \\ \hline 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\begin{pmatrix} X \\ Y \\ L_x \\ L_y \end{pmatrix} = \begin{pmatrix} 0 \\ 240 \\ 0 \\ 80 \end{pmatrix} + a \begin{pmatrix} 1 \\ -\frac{3}{4} \\ \frac{1}{4} \\ -\frac{1}{4} \end{pmatrix} \Rightarrow a = X \quad \Rightarrow \quad \begin{pmatrix} X \\ Y \\ L_x \\ L_y \end{pmatrix} = \begin{pmatrix} X \\ 240 - \frac{3}{4}X \\ \frac{1}{4}X \\ 80 - \frac{1}{4}X \end{pmatrix}$$

"in terms of" X

F57

(**Lecture** 10) S-8 **Free variables**

$$\begin{cases} x + 2y - z + w = -1 \\ -x - 2y + 3z + 5w = -5 \\ -x - 2y - z - 7w = 7 \end{cases}$$

1. Solve in terms of y and w
2. Solve in terms of x and w
3. Solve in terms of x and z
4. Solve in terms of x and y

F58

$$\left[\begin{array}{cccc|c} 1 & 2 & -1 & 1 & -1 \\ -1 & -2 & 3 & 5 & -5 \\ -1 & -2 & -1 & -7 & 7 \\ \hline 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{\substack{\boldsymbol{\tau} \\ [(-2)\mathbf{1}+\mathbf{2}] \\ [(1)\mathbf{1}+\mathbf{3}] \\ [(-1)\mathbf{1}+\mathbf{4}] \\ [(1)\mathbf{1}+\mathbf{5}]}} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 2 & 6 & -6 \\ -1 & 0 & -2 & -6 & 6 \\ \hline 1 & -2 & 1 & -1 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{\substack{\boldsymbol{\tau} \\ [(-3)\mathbf{3}+\mathbf{4}] \\ [(3)\mathbf{3}+\mathbf{5}]}} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 2 & 0 & 0 \\ -1 & 0 & -2 & 0 & 0 \\ \hline 1 & -2 & 1 & -4 & 4 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -3 & 3 \\ 0 & 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{cc|c} -2 & -4 & 4 \\ 1 & 0 & 0 \\ 0 & -3 & 3 \\ 0 & 1 & 0 \end{array}\right] \begin{cases} \xrightarrow{\substack{\boldsymbol{\tau} \\ [(\frac{-1}{2})\mathbf{1}] \\ [(4)\mathbf{1}+\mathbf{2}] \\ [(-4)\mathbf{1}+\mathbf{3}]}} \left[\begin{array}{cc|c} 1 & 0 & 0 \\ \frac{-1}{2} & -2 & 2 \\ 0 & -3 & 3 \\ 0 & 1 & 0 \end{array}\right] \xrightarrow{\substack{\boldsymbol{\tau} \\ [(1)\mathbf{2}+\mathbf{3}] \\ [(\frac{-1}{3})\mathbf{2}]}} \left[\begin{array}{cc|c} 1 & 0 & 0 \\ \frac{-1}{2} & \frac{2}{3} & 0 \\ 0 & 1 & 0 \\ 0 & \frac{-1}{3} & 1 \end{array}\right] \\ \xrightarrow{\substack{\boldsymbol{\tau} \\ [(\frac{-1}{2})\mathbf{1}] \\ [(4)\mathbf{1}+\mathbf{2}] \\ [(-4)\mathbf{1}+\mathbf{3}]}} \left[\begin{array}{cc|c} 1 & 0 & 0 \\ \frac{-1}{2} & -2 & 2 \\ 0 & -3 & 3 \\ 0 & 1 & 0 \end{array}\right] \xrightarrow{\substack{\boldsymbol{\tau} \\ [(\frac{-1}{2})\mathbf{2}] \\ [(\frac{1}{2})\mathbf{2}+\mathbf{1}] \\ [(-2)\mathbf{2}+\mathbf{3}]}} \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{3}{4} & \frac{3}{2} & 0 \\ \frac{-1}{4} & \frac{-1}{2} & 1 \end{array}\right] \end{cases}$$

The lecture ends here

Questions of the Optional Lecture 1

(L-Opt-1) Question 1.

(a) What is the smallest subspace of 3 by 3 matrices which contains all symmetric matrices and all lower triangular matrices?
(b) What is the largest subspace which is contained in both of those subspaces?

(Strang, 1988, exercise 4 from section 1.2.)

(L-Opt-1) Question 2. For each of these statements, say whether the claim is true or false and give a brief justification.

(a) **True/False:** The set of 3×3 non-invertible matrices forms a subspace of the set of all 3×3 matrices.
(b) **True/False:** If the system $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$ has no solution then $\mathbf{A}$ does not have full row rank.
(c) **True/False:** There exist $n \times n$ matrices $\mathbf{A}$ and $\mathbf{B}$ such that $\mathbf{B}$ is not invertible but $\mathbf{AB}$ is invertible.
(d) **True/False:** For any permutation matrix $\mathbf{P}$, we have that $\mathbf{P}^2 = \mathbf{I}$.

MIT Course 18.06 Quiz 1, October 4, 2004

(L-Opt-1) Question 3.

(a) Sean los vectores $\boldsymbol{u}$, $\boldsymbol{v}$ y $\boldsymbol{w}$ en $\mathbb{R}^7$ ¿Cuál es la dimensión (o cuáles son las posibles dimensiones) del espacio generado por estos tres vectores?
(b) Sea una matriz cuadrada $\mathbf{A}$. Si su espacio nulo $\mathcal{N}\left(\mathbf{A}\right)$ está compuesto únicamente por el vector nulo $\mathbf{0}$, ¿Cuál es el espacio nulo de su traspuesta (espacio nulo por la izquierda $\mathcal{N}\left(\mathbf{A}^\intercal\right)$?
(c) Piense en el espacio vectorial de todas las matrices de orden 5 por 5, $\mathbb{R}^{5\times5}$. Piense en el subconjunto de matrices 5 por 5 que son invertibles ¿es este subconjunto un sub-espacio vectorial? Si lo es, explique el motivo; si no lo es encuentre un contraejemplo.
(d) Indique si la siguiente aseveración es verdadera o falsa. Si es verdadera explique el motivo, si es falsa encuentre un contraejemplo: *"Si* $\mathbf{B}^2 = \mathbf{0}$, *entonces necesariamente* $\mathbf{B} = \mathbf{0}$*"*
(e) Si intercambio dos columnas de la matriz $\mathbf{A}$ ¿qué espacios fundamentales siguen siendo iguales?
(f) Si intercambio dos filas de la matriz $\mathbf{A}$ ¿qué espacios fundamentales siguen siendo iguales?

(g) ¿Por qué el vector $\boldsymbol{v} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ no puede estar en el espacio nulo de una matriz $\mathbf{A}$ y simultáneamente ser una fila de dicha matriz?

(L-Opt-1) Question 4. Empleando la definición de sub-espacio vectorial, verifique si los siguientes subconjuntos son sub-espacios vectoriales del espacio vectorial que los contiene.

(a) $\mathcal{V}$ es el espacio vectorial de todas las matrices 2×2 de números reales, con las operaciones habituales de suma y producto por un escalar; y el conjunto $\mathcal{W}$ son todas las matrices de la forma

$$\begin{bmatrix} a & b \\ 0 & b \end{bmatrix}$$

donde a y b son números reales.

(b) $\mathcal{V}$ es el espacio vectorial $C[0,1]$ de todas las funciones continuas en el intervalo $[0,1]$; y el conjunto $\mathcal{W}$ son todas las funciones $f \in C[0,1]$ tales que $f(0) = 2$.

(L-Opt-1) Question 5. Encuentre una base (de dimensión infinita) para el espacio de todos los polinomios

$$\mathcal{P} = \Big\{ a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 \mid \text{ para todo } n \Big\}.$$

(L-Opt-1) Question 6. ¿Cuál es la dimensión de los siguientes espacios?

(a) El conjunto de matrices simétricas de orden 2×2, $\mathbf{A} = \mathbf{A}^\intercal$.

$$\mathbf{A} = \begin{bmatrix} a & b \\ b & d \end{bmatrix},$$

(b) El conjunto de matrices simétricas de orden 2×2

$$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

tales que $a + d = 0$.

(c) El conjunto de vectores de $\mathbb{R}^4$ de la forma $\Big\{ \Big(x,\ y,\ (x-3y),\ (2y-x), \Big) \ \mid\ x, y \in \mathbb{R} \Big\}$.

End of Questions of the Optional Lecture 1

References

Strang, G. (1988). *Linear algebra and its applications*. Thomson Learning, Inc., third ed. ISBN 0-15-551005-3.

Strang, G. (2003). *Introduction to Linear Algebra*. Wellesley-Cambridge Press, Wellesley, Massachusetts. USA, third ed. ISBN 0-9614088-9-8.

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Solutions

(L-6) Question 1(a) All points in the Quadrant I of $\mathbb{R}^2$, that is, $\{(x, y) : x \geq 0, y \geq 0\}$. It is close under addition, but not under multiplication by an escalar $c\boldsymbol{v}$ when $c < 0$. What about the other quadrants?

□

(L-6) Question 1(b) Take the union of two lines that goes through the origin, for example $\{(x, y) : x = y\}$ and $\{(x, y) : x = -y\}$. This set of points is close under multiplication by a scalar, since any multiple of any point is always in one line. But it isn't close under addition. What about the first and third quadrants together?

□

(L-6) Question 2(a) It is not a subspace. The addition $(1, 1) + (2, 4) = (3, 5)$ doesn't belong to the set; there fore, this set is not close under addition

□

(L-6) Question 2(b) It is a subspace

□

(L-6) Question 2(c) It is a subspace

□

(L-6) Question 2(d) It is not a subspace. It is not closed under multiplication. Consider for example $\frac{1}{2} \cdot (1, 1)$.

□

(L-6) Question 2(e) It is not a subspace. It is not closed under addition. Consider for example $(1, 0) + (0, 1)$.

□

(L-6) Question 2(f) It is a subspace

□

(L-6) Question 3. $\mathbb{R}^2$ contains vectors with *two* components —they don't belong to $\mathbb{R}^3$.

□

(L-6) Question 4. Vectors $(1, 0, -2,)$ and $(0, 1, -4,)$ belong to P, but their addition, $(1, 1, -6,)$, doesn't belong to P, since $1 - 1 - (-6) = 6 \neq 3$.

□

(L-6) Question 5. Assume (wrongly) that the set of solutions of $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$, with $\boldsymbol{b} \neq \mathbf{0}$, is a subspace. Then, for any couple of solutions $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$, its addition should be also a solution, and then $\mathbf{A}(\boldsymbol{x}_1 + \boldsymbol{x}_2) = \boldsymbol{b}$, but also $\mathbf{A}(\boldsymbol{x}_1 + \boldsymbol{x}_2) = \mathbf{A}\boldsymbol{x_1} + \mathbf{A}\boldsymbol{x_2} = \boldsymbol{b} + \boldsymbol{b}$; and therefore $\boldsymbol{b} = \boldsymbol{b} + \boldsymbol{b}$; something that contradicts our assumption $\boldsymbol{b} \neq \mathbf{0}$.

Therefore the solution set $\{\boldsymbol{x} \mid \mathbf{A}\boldsymbol{x} = \boldsymbol{b}\}$ does not form a subspace.

□

(L-6) Question 6(a) All the multiples of $\mathbf{A}$

$$c\mathbf{A}$$

form a subspace, and $\mathbf{B}$ does not belong to it.

Also the set of matrices:

$$\left\{\mathbf{M} \in \mathbb{R}^{2\times 2} \,\middle|\, \exists a, b, c \in \mathbb{R}, \mathbf{M} = \begin{bmatrix} a & b \\ c & 0 \end{bmatrix}\right\}$$

□

(L-6) Question 6(b) All the multiples of $\mathbf{B}$

$$c\mathbf{B}$$

form a subspace, and $\mathbf{A}$ does not belong to it.

Also the set of matrices:

$$\left\{\mathbf{M} \in \mathbb{R}^{2\times 2} \,\middle|\, \exists f, g, h \in \mathbb{R}, \mathbf{M} = \begin{bmatrix} 0 & f \\ g & h \end{bmatrix}\right\}$$

□

(L-6) Question 6(c) No. All subspaces are closed under linear combinations, therefore, if it contains $\mathbf{A}$ and $\mathbf{B}$, then it must contain $\frac{1}{2}\mathbf{A} - \frac{1}{3}\mathbf{B} = \mathbf{I}$.

□

(L-6) Question 7(a) It is a subspace, since any linear combination of matrices in $\mathcal{S}$ is another symmetric matrix: $_{j|}\Big(\big(a\mathbf{A}+b\mathbf{B}\big)^{\intercal}\Big) = \big(a\mathbf{A}+b\mathbf{B}\big)_{|j} = a\big(\mathbf{A}_{|j}\big) + b\mathbf{B}_{|j} = a\big(_{j|}\mathbf{A}\big) + b_{j|}\mathbf{B} = {}_{j|}\big(a\mathbf{A}+b\mathbf{B}\big).$

□

(L-6) Question 7(b) It is NOT a subspace. It isn't close under addition (it is easy to find two non-symmetric matrices whose addition is a symmetric one). For example:

$$\mathbf{A} = \begin{bmatrix}1 & 3\\ 0 & 1\end{bmatrix}; \qquad \mathbf{B} = \begin{bmatrix}1 & 0\\ 3 & 1\end{bmatrix}; \qquad \mathbf{A}+\mathbf{B} = \begin{bmatrix}2 & 3\\ 3 & 2\end{bmatrix}$$

In fact, for any $\mathbf{A}$ always $(\mathbf{A} + \mathbf{A}^{\intercal})$ is a symmetric matrix.

□

(L-6) Question 7(c) It is a subspace, since any linear combination

$$c\begin{bmatrix}0 & -a_{12} & \dots & -a_{1m}\\ a_{12} & 0 & \dots & -a_{2m}\\ \vdots & \vdots & \ddots & \vdots\\ a_{1m} & a_{2m} & \dots & 0\end{bmatrix} + d\begin{bmatrix}0 & -b_{12} & \dots & -b_{1m}\\ b_{12} & 0 & \dots & -b_{2m}\\ \vdots & \vdots & \ddots & \vdots\\ b_{1m} & b_{2m} & \dots & 0\end{bmatrix} = \begin{bmatrix}0 & -ca_{12}-db_{12} & \dots & -ca_{1m}-db_{1m}\\ ca_{12}+db_{12} & 0 & \dots & -ca_{2m}-db_{2m}\\ \vdots & \vdots & \ddots & \vdots\\ ca_{1m}+db_{1m} & ca_{2m}+db_{2m} & \dots & 0\end{bmatrix}$$

is another skew-symmetric matrix.

Alternatively; since

$$\mathbf{A}^{\intercal} = -\mathbf{A} \quad \Leftrightarrow \quad \mathbf{A}_{|j} = \big(-\mathbf{A}^{\intercal}\big)_{|j} = {}_{j|}\big(-\mathbf{A}\big)$$

then $_{j|}\Big(\big(a\mathbf{A}+b\mathbf{B}\big)^{\intercal}\Big) = \big(a\mathbf{A}+b\mathbf{B}\big)_{|j} = \big(a\mathbf{A}\big)_{|j} + \big(b\mathbf{B}\big)_{|j} = {}_{j|}\big(-a\mathbf{A}\big) + {}_{j|}\big(-b\mathbf{B}\big) = {}_{j|}\Big(-\big(a\mathbf{A}+b\mathbf{B}\big)\Big).$

□

(L-6) Question 8(a) A line. A plane.

□

(L-6) Question 8(b) A point. A line

□

(L-6) Question 8(c) We should check that $\boldsymbol{x}+\boldsymbol{y} \in \mathcal{S}\cap\mathcal{T}$ and $c\boldsymbol{x} \in \mathcal{S}\cap\mathcal{T}$

Since $\boldsymbol{x}$ and $\boldsymbol{y}$ belong to $\mathcal{S}\cap\mathcal{T}$, both belong to $\mathcal{S}$, and both belong to $\mathcal{T}$. Therefore,

$$\begin{aligned} \boldsymbol{x}+\boldsymbol{y} \in \mathcal{S} &\quad \text{since } \mathcal{S} \text{ is a subspace and } \boldsymbol{x},\boldsymbol{y} \in \mathcal{S}\\ \boldsymbol{x}+\boldsymbol{y} \in \mathcal{T} &\quad \text{since } \mathcal{T} \text{ is a subspace and } \boldsymbol{x},\boldsymbol{y} \in \mathcal{T}\end{aligned}$$

therefore the addition belongs to the intersection of both subspaces. Besides

$$\begin{aligned} c\boldsymbol{x} \in \mathcal{S} &\quad \text{since } \mathcal{S} \text{ is a subspace and } \boldsymbol{x} \in \mathcal{S}\\ c\boldsymbol{x} \in \mathcal{T} &\quad \text{since } \mathcal{T} \text{ is a subspace and } \boldsymbol{x} \in \mathcal{T}\end{aligned}$$

therefore $c\boldsymbol{x}$ belongs to the intersection of both subspaces.

□

(L-6) Question 9(a) It is a subspace, since any linear combination of two of those vectors also has a zero in the first component

□

(L-6) Question 9(b) No, it isn't a subspace. If we add two of them we get a vector with first component $b_1 = 2$. Also, the zero vector $\mathbf{0}$ doesn't belong top the set.

□

(L-6) Question 9(c) No, it isn't a subspace. If we add $(1,0,1,)$ (from the first plane) and $(1,1,0,)$ (from the second) we get a vector outside both planes.

□

(L-6) Question 9(d) It is a subspace, since any linear combination of zero vectors $a\mathbf{0} + b\mathbf{0} = \mathbf{0}$. is inside the set $\{\mathbf{0}\}$ (the solitary vector $\mathbf{0}$)

□

(L-6) Question 9(e) Yes, it is by construction.

□

(L-6) Question 9(f) It is a subspace, since for any linear combination

$$\boldsymbol{v} = a\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} + c\begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix} = \begin{pmatrix} ab_1 + cd_1 \\ ab_2 + cd_2 \\ ab_3 + cd_3 \end{pmatrix}$$

then

$$(ab_3 + cd_3) - (ab_2 + cd_2) + 3(ab_1 + cd_1) = a(b_3 - b_2 + 3b_1) + c(d_3 - d_2 + 3d_1) = a \cdot 0 + c \cdot 0 = 0$$

which proves that the new vector also satisfy the condition $v_3 - v_2 + 3v_1 = 0$.

□

(L-6) Question 10(a)

1. $\boldsymbol{x} + \boldsymbol{y} = \Big(x_1,\ x_2,\Big) + \Big(y_1,\ y_2,\Big) = \Big((x_1 + y_1 + 1),\ (x_2 + y_2 + 1),\Big) = \Big((y_1 + x_1 + 1),\ (y_2 + x_2 + 1),\Big) = \Big(y_1,\ y_2,\Big) + \Big(x_1,\ x_2,\Big) = \boldsymbol{y} + \boldsymbol{x}$
2. $$\boldsymbol{x} + (\boldsymbol{y} + \boldsymbol{z}) = \Big(x_1,\ x_2,\Big) + \Big((y_1 + z_1 + 1),\ (y_2 + z_2 + 1),\Big) = \Big((x_1 + y_1 + z_1 + 2),\ (x_2 + y_2 + z_2 + 2),\Big) =$$
   $$= \Big((x_1 + y_1 + 1),\ (x_2 + y_2 + 1),\Big) + \Big(z_1 + z_2\Big) = (\boldsymbol{x} + \boldsymbol{y}) + \boldsymbol{z}$$
3. The rule is unbroken, but the new "zero vector" $\mathbf{0}$ in this case is: $\mathbf{0} = (-1,\ -1,)$ instead of $(0,\ 0,)$; since
   $$\boldsymbol{x} + \mathbf{0} = (x_1,\ x_2) + (-1,\ -1) = \Big((x_1 - 1 + 1),\ (x_2 - 1 + 1),\Big) = (x_1, x_2) = \boldsymbol{x}.$$
4. This rule is unbroken, although if $\boldsymbol{x} = (x_1, x_2,)$ then $-\boldsymbol{x}$ should be $-\boldsymbol{x} = \Big((-x_1 - 2),\ (x_2 - 2),\Big)$ since
   $$\boldsymbol{x} + (-\boldsymbol{x}) = (x_1,\ x_2,) + \Big((-x_1 - 2),\ (x_2 - 2),\Big) = (-1,\ -1,) = \mathbf{0}.$$
   note that here the third rule implies $\mathbf{0} = (-1,\ -1,)$.
5. $1\boldsymbol{x} = 1(x_1,\ x_2,) = (1x_1,\ 1x_2,) = \boldsymbol{x}$
6. $ab\boldsymbol{x} = ab(x_1,\ x_2,) = (abx_1,\ abx_2,) = a(bx_1,\ bx_2,) = a(b\boldsymbol{x})$.
7. The rule $a(\boldsymbol{x} + \boldsymbol{y}) = a\boldsymbol{x} + a\boldsymbol{y}$ **is broken**... for example
   $$2\Big((1,\ 1,) + (1,\ 1,)\Big) = 2(3,\ 3,) = (6,\ 6,) \neq 2(1,\ 1,) + 2(1,\ 1,) = (2,\ 2,) + (2,\ 2,) = (5,\ 5,).$$
8. The rule $(a + b)\boldsymbol{x} = a\boldsymbol{x} + b\boldsymbol{x}$ **is also broken**... for example
   $$(2 + 2)(1,\ 1,) = 4(1,\ 1,) = (4,\ 4,) \neq 2(1,\ 1,) + 2(1,\ 1,) = (2,\ 2,) + (2,\ 2,) = (5,\ 5,).$$

□

(L-6) Question 10(b)

1. $\boldsymbol{x} + \boldsymbol{y} = xy = yx = \boldsymbol{y} + \boldsymbol{x}$.
2. $\boldsymbol{x} + (\boldsymbol{y} + \boldsymbol{z}) = x(yz) = (xy)z = (\boldsymbol{x} + \boldsymbol{y}) + \boldsymbol{z}$.
3. If $\mathbf{0} = 1$; then $\boldsymbol{x} + \mathbf{0} = x1 = x = \boldsymbol{x}$;.

4. If $-\boldsymbol{x} = 1/x$; then $\quad \boldsymbol{x} + (-\boldsymbol{x}) = x/x = 1 = \mathbf{0}$.
5. $1\boldsymbol{x} = x^1 = x = \boldsymbol{x}$.
6. $(ab)\boldsymbol{x} = x^{(ab)} = (x^b)^a = (b\boldsymbol{x})^a = a(b\boldsymbol{x})$.
7. $a(\boldsymbol{x} + \boldsymbol{y}) = (xy)^a = x^a \cdot y^a = a\boldsymbol{x} + a\boldsymbol{y}$.
8. $(a + b)\boldsymbol{x} = (x)^{a+b} = x^a \cdot x^b = a\boldsymbol{x} + b\boldsymbol{x}$.

$\square$

(L-6) Question 10(c)

1. $\boldsymbol{x} + \boldsymbol{y} = \Big((x_1 + y_2),\ (x_2 + y_1),\Big) \neq \Big((y_1 + x_2),\ (y_2 + x_1),\Big) = \boldsymbol{y} + \boldsymbol{x}$. It is not satisfied. For example $(-1,\ 1,) + (2,\ 3,) = \Big((-1 + 3),\ (1 + 2),\Big) = (2,\ 3,) \neq (3,\ 2,) = \Big((2 + 1),\ (3 - 1),\Big) = (2,\ 3,) + (-1,\ 1,)$
2. $\boldsymbol{x} + (\boldsymbol{y} + \boldsymbol{z}) = (\boldsymbol{x} + \boldsymbol{y}) + \boldsymbol{z}$. It is not satisfied. For example $(0,\ 0,) + \Big((-1,\ 1,) + (2,\ 3,)\Big) = (0,\ 0,) + (2,\ 3,) = (3,\ 2,) \neq (4,\ 1,) = (1,\ -1,) + (2,\ 3,) = \Big((0,\ 0,) + (-1,\ 1,)\Big) + (2,\ 3,)$
3. $\boldsymbol{x} + \mathbf{0} = (x_1,\ x_2,) + (0,\ 0,) = \Big((x_1 + 0),\ (x_2 + 0),\Big) = \boldsymbol{x}$;
4. If $-\boldsymbol{x} = (-x_2,\ -x_1,)$; then $\quad \boldsymbol{x} + (-\boldsymbol{x}) = (x_1,\ x_2,) + (-x_2,\ -x_1,) = (0,\ 0,) = \mathbf{0}$.
5. $1\boldsymbol{x} = 1(x_1,\ x_2,) = \boldsymbol{x}$.
6. $(ab)\boldsymbol{x} = (abx_1,\ abx_2,) = a(bx_1,\ bx_2,) = a(b\boldsymbol{x})$.
7. $a(\boldsymbol{x} + \boldsymbol{y}) = a\Big((x_1 + y_2),\ (x_2 + y_1),\Big) = (ax_1,\ ax_2,) + (ay_1,\ ay_2,) = a\boldsymbol{x} + a\boldsymbol{y}$.
8. It is not satisfied: $(a+b)\boldsymbol{x} = \Big((a+b)x_1,\ (a+b)x_2,\Big) = \Big((ax_1+bx_1),\ (ax_2+bx_2),\Big) \neq \Big((ax_1+bx_2),\ (ax_2+bx_1),\Big) = (ax_1,\ ax_2,) + (bx_1,\ bx_2,) = a\boldsymbol{x} + b\boldsymbol{x}$.

$\square$

(L-7) Question 1(a)

$$\left[\begin{array}{rrrr} 1 & 2 & 0 & 5 \\ 2 & 3 & 1 & 4 \\ -1 & -1 & -1 & 1 \\ \hline 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow[{[(-5)\mathbf{1}+\mathbf{4}]}]{\substack{\boldsymbol{\tau} \\ [(-2)\mathbf{1}+\mathbf{2}]}} \left[\begin{array}{rrrr} 1 & 0 & 0 & 0 \\ 2 & -1 & 1 & -6 \\ -1 & 1 & -1 & 6 \\ \hline 1 & -2 & 0 & -5 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow[{[(-6)\mathbf{2}+\mathbf{4}]}]{\substack{\boldsymbol{\tau} \\ [(1)\mathbf{2}+\mathbf{3}]}} \left[\begin{array}{rrrr} 1 & 0 & 0 & 0 \\ 2 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ \hline 1 & -2 & -2 & 7 \\ 0 & 1 & 1 & -6 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

Two pivots: rank 2. The null space,

$$\mathcal{N}\left(\mathbf{A}\right) = \left\{ \boldsymbol{v} \in \mathbb{R}^4 \ \middle|\ \exists \boldsymbol{p} \in \mathbb{R}^2,\ \boldsymbol{v} = \begin{bmatrix} -2 & 7 \\ 1 & -6 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \boldsymbol{p} \right\},$$

is a plane in a 4 dimensional space.

```
A = Matrix([[1,2,0,5],[2,3,1,4],[-1,-1,-1,1]])
Homogenea(A,1)
Math( SubEspacio(A).EcParametricas() )
```

$\square$

(L-7) Question 1(b)

$$\left[\begin{array}{ccc}1&2&1\\-1&3&4\\2&-1&-3\\\hline 1&0&0\\0&1&0\\0&0&1\end{array}\right]\xrightarrow[{[(-1)\mathbf{1}+\mathbf{3}]}]{\substack{\boldsymbol{\tau}\\{[(-2)\mathbf{1}+\mathbf{2}]}}}\left[\begin{array}{ccc}1&0&0\\-1&5&5\\2&-5&-5\\\hline 1&-2&-1\\0&1&0\\0&0&1\end{array}\right]\xrightarrow{\substack{\boldsymbol{\tau}\\{[(-1)\mathbf{2}+\mathbf{3}]}}}\left[\begin{array}{ccc}1&0&0\\-1&5&0\\2&-5&0\\\hline 1&-2&1\\0&1&-1\\0&0&1\end{array}\right]=\left[\frac{\mathsf{K}}{\mathsf{E}}\right]$$

Two pivots: rank 2. It is a line in a 3 dimensional space.

$$\text{The null space,}\quad \mathcal{N}\left(\mathsf{F}\right)=\left\{\boldsymbol{v}\in\mathbb{R}^3\;\middle|\;\exists\boldsymbol{p}\in\mathbb{R}^1,\ \boldsymbol{v}=\begin{bmatrix}1\\-1\\1\end{bmatrix}\boldsymbol{p}\right\}.$$

```
F = Matrix([[1,2,1],[-1,3,4],[2,-1,-3]])
Homogenea(F,1)
Math( SubEspacio(F).EcParametricas() )
```

$\square$

(L-7) Question 1(c)

$$\left[\begin{array}{ccc}1&2&1\\0&3&1\\-2&-1&4\\\hline 1&0&0\\0&1&0\\0&0&1\end{array}\right]\xrightarrow[{[(-1)\mathbf{1}+\mathbf{3}]}]{\substack{\boldsymbol{\tau}\\{[(-2)\mathbf{1}+\mathbf{2}]}}}\left[\begin{array}{ccc}1&0&0\\0&3&1\\-2&3&6\\\hline 1&-2&-1\\0&1&0\\0&0&1\end{array}\right]\xrightarrow[{[(-1)\mathbf{2}+\mathbf{3}]}]{\substack{\boldsymbol{\tau}\\{[(3)\mathbf{3}]}}}\left[\begin{array}{ccc}1&0&0\\0&3&0\\-2&3&15\\\hline 1&-2&-1\\0&1&-1\\0&0&3\end{array}\right]=\left[\frac{\mathsf{K}}{\mathsf{E}}\right]$$

Three pivots: rank 3.

$$\text{The null space,}\quad \mathcal{N}\left(\mathsf{A}\right)=\left\{\begin{pmatrix}0\\0\\0\end{pmatrix}\right\}\quad\text{is a point in a 3 dimensional space.}$$

```
G = Matrix([[1,2,1],[0,3,1],[-2,-1,4]])
Homogenea(G,1)
```

$\square$

(L-7) Question 1(d)

$$\left[\frac{\mathsf{H}}{\mathsf{I}}\right]=\left[\begin{array}{cc}1&3\\2&1\\-1&-3\\\hline 1&0\\0&1\end{array}\right]\xrightarrow{\substack{\boldsymbol{\tau}\\{[(-3)\mathbf{1}+\mathbf{2}]}}}\left[\begin{array}{cc}1&0\\2&-5\\-1&0\\\hline 1&-3\\0&1\end{array}\right]=\left[\frac{\mathsf{K}}{\mathsf{E}}\right]$$

Two pivots: rank 2. There aren't zero columns.

$$\text{The null space,}\quad \mathcal{N}\left(\mathsf{H}\right)=\left\{\begin{pmatrix}0\\0\end{pmatrix}\right\}\quad\text{is a point in a 2 dimensional space.}$$

```
H = Matrix([[1,3],[2,1],[-1,-3]])
Homogenea(H,1)
```

$\square$

(L-7) Question 2(a)

$$\left[\begin{array}{ccc} 1 & 2 & -2 \\ 2 & 1 & 0 \\ \hline 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right] \xrightarrow[\substack{[(-2)\mathbf{1}+\mathbf{2}] \\ [(2)\mathbf{1}+\mathbf{3}]}]{\tau} \left[\begin{array}{ccc} 1 & 0 & 0 \\ 2 & -3 & 4 \\ \hline 1 & -2 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right] \xrightarrow[\substack{[(3)\mathbf{3}] \\ [(4)\mathbf{2}+\mathbf{3}]}]{\tau} \left[\begin{array}{ccc} 1 & 0 & 0 \\ 2 & -3 & 0 \\ \hline 1 & -2 & -2 \\ 0 & 1 & 4 \\ 0 & 0 & 3 \end{array}\right]$$

Therefore $\mathcal{N}\left(\mathbf{A}\right) = \left\{ \boldsymbol{v} \in \mathbb{R}^3 \;\middle|\; \exists \boldsymbol{p} \in \mathbb{R}^1,\ \boldsymbol{v} = \begin{bmatrix} -2 \\ 4 \\ 3 \end{bmatrix} \boldsymbol{p} \right\}$ is a line in a 3 dimensional space.

□

(L-7) Question 2(b) All are pivot columns, then the sole solution to $\mathbf{F}\boldsymbol{x} = \mathbf{0}$ is the zero vector, therefore $\mathcal{N}\left(\mathbf{F}\right) = \{\mathbf{0}\}$. A point in $\mathbb{R}^2$.

□

(L-7) Question 2(c)

$$\left[\begin{array}{ccc} 1 & 2 & -4 \\ -1 & 1 & 3 \\ 1 & 5 & -5 \\ \hline 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right] \xrightarrow[\substack{[(-2)\mathbf{1}+\mathbf{2}] \\ [(4)\mathbf{1}+\mathbf{3}]}]{\tau} \left[\begin{array}{ccc} 1 & 0 & 0 \\ -1 & 3 & -1 \\ 1 & 3 & -1 \\ \hline 1 & -2 & 4 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right] \xrightarrow[\substack{[(3)\mathbf{3}] \\ [(1)\mathbf{2}+\mathbf{3}]}]{\tau} \left[\begin{array}{ccc} 1 & 0 & 0 \\ -1 & 3 & 0 \\ 1 & 3 & 0 \\ \hline 1 & -2 & 10 \\ 0 & 1 & 1 \\ 0 & 0 & 3 \end{array}\right]$$

Therefore $\mathcal{N}\left(\mathbf{G}\right) = \left\{ \boldsymbol{v} \in \mathbb{R}^3 \;\middle|\; \exists \boldsymbol{p} \in \mathbb{R}^1,\ \boldsymbol{v} = \begin{bmatrix} 10 \\ 1 \\ 3 \end{bmatrix} \boldsymbol{p} \right\}$ is a line in a $\mathbb{R}^3$.

□

(L-7) Question 3.

$$\left[\begin{array}{cccc} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \\ \hline 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow[\substack{[(-2)\mathbf{1}+\mathbf{2}] \\ [(-1)\mathbf{1}+\mathbf{4}]}]{\tau} \left[\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ \hline 1 & -2 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow[{[(-1)\mathbf{2}+\mathbf{3}]}]{\tau} \left[\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ \hline 1 & -2 & 2 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

Then, rank is 2. x_1 and x_2 are pivot variables.

The special solutions are

$$\boldsymbol{x}_a = \begin{pmatrix} 2 \\ -1 \\ 1 \\ 0 \end{pmatrix}; \qquad \boldsymbol{x}_b = \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

Finally, the complete solution to $\mathbf{A}\boldsymbol{x} = \mathbf{0}$ is

$$\left\{ \boldsymbol{v} \in \mathbb{R}^4 \;\middle|\; \exists \boldsymbol{p} \in \mathbb{R}^2,\ \boldsymbol{v} = \begin{bmatrix} 2 & -1 \\ -1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \boldsymbol{p} \right\}$$

□

(L-7) Question 4(a)

$$\left[\frac{\mathbf{A}}{\mathbf{I}}\right] = \left[\begin{array}{cccc} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ \hline 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow[\substack{[(-1)\mathbf{1}+\mathbf{2}] \\ [(-1)\mathbf{1}+\mathbf{3}] \\ [(-1)\mathbf{1}+\mathbf{4}]}]{\boldsymbol{\tau}} \left[\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ \hline 1 & -1 & -1 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right] = \left[\frac{\mathbf{L}}{\mathbf{E}}\right]$$

Rank 1. Pivot variable: x_1. The complete solution is

$$\mathcal{N}\left(\mathbf{A}\right) = \left\{ \boldsymbol{x} \in \mathbb{R}^4 \,\middle|\, \text{exists } \begin{pmatrix} a \\ b \\ c \end{pmatrix} \in \mathbb{R}^3 \text{ such that } \boldsymbol{x} = \begin{bmatrix} -1 & -1 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} -a-b-c \\ a \\ b \\ c \end{pmatrix} \right\}$$

$\square$

(L-7) Question 4(b)

$$\left[\begin{array}{cccc} -1 & 1 & -1 & 1 \\ 1 & 1 & 1 & 1 \\ -1 & 1 & -1 & 1 \\ \hline 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow[\substack{[(1)\mathbf{1}+\mathbf{2}] \\ [(-1)\mathbf{1}+\mathbf{3}] \\ [(1)\mathbf{1}+\mathbf{4}]}]{\boldsymbol{\tau}} \left[\begin{array}{cccc} -1 & 0 & 0 & 0 \\ 1 & 2 & 0 & 2 \\ -1 & 0 & 0 & 0 \\ \hline 1 & 1 & -1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow[{[(-1)\mathbf{2}+\mathbf{4}]}]{\boldsymbol{\tau}} \left[\begin{array}{cccc} -1 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ \hline 1 & 1 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right] = \left[\frac{\mathbf{K}}{\mathbf{E}}\right]$$

Rank 2. Pivot variables: x_1 and x_2. The complete solution is

$$\left\{ \boldsymbol{x} \in \mathbb{R}^4 \,\middle|\, \text{exists } \begin{pmatrix} a \\ b \end{pmatrix} \in \mathbb{R}^2 \text{ such that } \boldsymbol{x} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} -a \\ -b \\ a \\ b \end{pmatrix} \right\}$$

$\square$

(L-7) Question 4(c)

$$\left[\frac{\mathbf{A}}{\mathbf{I}}\right] = \left[\begin{array}{cccc} -1 & 1 & -1 & 1 \\ -1 & 1 & -1 & 1 \\ -1 & 1 & -1 & 1 \\ \hline 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow[\substack{[(1)\mathbf{1}+\mathbf{2}] \\ [(-1)\mathbf{1}+\mathbf{3}] \\ [(1)\mathbf{1}+\mathbf{4}]}]{\boldsymbol{\tau}} \left[\begin{array}{cccc} -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ \hline 1 & 1 & -1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right] = \left[\frac{\mathbf{K}}{\mathbf{E}}\right]$$

Rank 1. Pivot variable: x_1. The complete solution is

$$\left\{ \boldsymbol{x} \in \mathbb{R}^4 \,\middle|\, \text{exists } \begin{pmatrix} a \\ b \\ c \end{pmatrix} \in \mathbb{R}^3 \text{ such that } \boldsymbol{x} = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} a-b+c \\ a \\ b \\ c \end{pmatrix} \right\}$$

$\square$

(L-7) Question 5(a) Since $\boldsymbol{x}_1, \boldsymbol{x}_2 \in \mathbb{R}^3$, $\mathbf{A}$ has 3 columns.

$\square$

(L-7) Question 5(b) Any number greater (or equal) than one.

$\square$

(L-7) Question 5(c) Three columns and two special solutions (2 free columns) means rank 1 (only one pivot column).

□

(L-7) Question 6(a) Since **R** only has two pivots, we know the third column is a linear combination of the other two (it is a free column).

□

(L-7) Question 6(b) By column Gaussian elimination

$$\begin{bmatrix}1&2\\2&a\\1&1\\b&8\end{bmatrix}\xrightarrow{\substack{\boldsymbol{\tau}\\[(-2)\mathbf{1}+\mathbf{2}]}}\begin{bmatrix}1&0\\2&a-4\\1&-1\\b&8-2b\end{bmatrix}\xrightarrow[a=4]{\substack{\boldsymbol{\tau}\\[(1)\mathbf{2}+\mathbf{1}]}}\begin{bmatrix}1&0\\2&0\\0&-1\\8-b&8-2b\end{bmatrix}\xrightarrow{\substack{\boldsymbol{\tau}\\[(-1)\mathbf{2}]}}\begin{bmatrix}1&0\\2&0\\0&1\\3&2\end{bmatrix}$$

And since $8-b=3$, then $b=5$.

□

(L-7) Question 6(c) Since there is only one free column, we know that $\mathbf{A}\begin{pmatrix}1\\-2\\1\end{pmatrix}=\mathbf{0}$ then

$$\dim(\mathcal{N}(\mathbf{A}))=1;\qquad \mathcal{N}(\mathbf{A})=\left\{\boldsymbol{x}\in\mathbb{R}^3\;\middle|\;\text{exists } c\in\mathbb{R}\text{such that } \boldsymbol{x}=c\begin{pmatrix}1\\-2\\1\end{pmatrix}\right\}$$

```
a=Vector([1,-2,1])
SubEspacio(Sistema([a]))
```

□

(L-7) Question 7(a)

$$\begin{bmatrix}1&1&1&1\\1&1&1&1\\0&1&2&3\\0&1&2&3\end{bmatrix}\xrightarrow{\substack{\boldsymbol{\tau}\\[(-1)\mathbf{1}+\mathbf{2}]\\[(-1)\mathbf{1}+\mathbf{3}]\\[(-1)\mathbf{1}+\mathbf{4}]}}\begin{bmatrix}1&0&0&0\\1&0&0&0\\0&1&2&3\\0&1&2&3\end{bmatrix}\xrightarrow{\substack{\boldsymbol{\tau}\\[(-2)\mathbf{2}+\mathbf{3}]\\[(-3)\mathbf{2}+\mathbf{4}]}}\begin{bmatrix}1&0&0&0\\1&0&0&0\\0&1&0&0\\0&1&0&0\end{bmatrix}$$

```
A = Matrix([[1,1,1,1],[1,1,1,1],[0,1,2,3],[0,1,2,3]])
R = ElimGJ(A,1)
```

□

(L-7) Question 7(b) $\begin{bmatrix}1&2&1\\2&2&2\\1&0&1\end{bmatrix}\xrightarrow{\substack{\boldsymbol{\tau}\\[(-2)\mathbf{1}+\mathbf{2}]\\[(-1)\mathbf{1}+\mathbf{3}]}}\begin{bmatrix}1&0&0\\2&-2&0\\1&-2&0\end{bmatrix}\xrightarrow{\substack{\boldsymbol{\tau}\\[(1)\mathbf{2}+\mathbf{1}]}}\begin{bmatrix}1&0&0\\0&-2&0\\-1&-2&0\end{bmatrix}\xrightarrow{\substack{\boldsymbol{\tau}\\[(-\frac{1}{2})\mathbf{2}]}}\begin{bmatrix}1&0&0\\0&1&0\\-1&1&0\end{bmatrix}$

```
B = Matrix([[1,2,1],[2,2,2],[1,0,1]])
R = ElimGJ(B,1)
```

□

(L-7) Question 7(c) $\begin{bmatrix}1&2&1&2&1\\2&1&2&1&2\\0&1&0&1&0\end{bmatrix}\xrightarrow{\substack{\boldsymbol{\tau}\\[(-2)\mathbf{1}+\mathbf{2}]\\[(-1)\mathbf{1}+\mathbf{3}]\\[(-2)\mathbf{1}+\mathbf{4}]\\[(-1)\mathbf{1}+\mathbf{5}]}}\begin{bmatrix}1&0&0&0&0\\2&-3&0&-3&0\\0&1&0&1&0\end{bmatrix}\xrightarrow{\substack{\boldsymbol{\tau}\\[(3)\mathbf{1}]\\[(2)\mathbf{2}+\mathbf{1}]\\[(-1)\mathbf{2}+\mathbf{4}]}}\begin{bmatrix}3&0&0&0&0\\0&-3&0&0&0\\2&1&0&0&0\end{bmatrix}\xrightarrow{\substack{\boldsymbol{\tau}\\[(\frac{1}{3})\mathbf{1}]\\[(-\frac{1}{3})\mathbf{2}]}}$
$\begin{bmatrix}1&0&0&0&0\\0&1&0&0&0\\\frac{2}{3}&-\frac{1}{3}&0&0&0\end{bmatrix}$

(L-7) Question 7(d) $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \end{bmatrix} \xrightarrow{\substack{\tau \\ [(-2)\mathbf{1}+\mathbf{2}] \\ [(-3)\mathbf{1}+\mathbf{3}]}} \begin{bmatrix} 1 & 0 & 0 \\ 2 & -1 & -2 \end{bmatrix} \xrightarrow{\substack{\tau \\ [(2)\mathbf{2}+\mathbf{1}] \\ [(-2)\mathbf{2}+\mathbf{3}]}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix} \xrightarrow{\substack{\tau \\ [(-1)\mathbf{2}]}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

(L-7) Question 8(a) The identity matrix.

(L-7) Question 8(b)

$$\left[\begin{array}{c} \mathsf{A} \\ \hline \mathsf{I} \end{array}\right] = \left[\begin{array}{ccc} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \\ \hline 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right] \xrightarrow{\substack{\tau \\ [(-1)\mathbf{2}+\mathbf{1}] \\ [(-1)\mathbf{2}+\mathbf{3}]}} \left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ \hline 1 & 0 & 0 \\ -1 & 1 & -1 \\ 0 & 0 & 1 \end{array}\right] = \left[\begin{array}{c} \mathsf{I} \\ \hline \mathsf{E} \end{array}\right]. \text{ Therefore } \mathsf{A}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

```
A = Matrix([[1,0,0],[1,1,1],[0,0,1]])
InvMat(A,1)
```

(L-7) Question 9(a) The identity matrix I.

(L-7) Question 9(b)

$$\left[\begin{array}{c} \mathsf{A} \\ \hline \mathsf{I} \end{array}\right] = \left[\begin{array}{ccc} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \\ \hline 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right] \xrightarrow{\substack{\tau \\ [\mathbf{1}\rightleftharpoons\mathbf{3}]}} \left[\begin{array}{ccc} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \\ \hline 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{array}\right] \xrightarrow{\substack{\tau \\ [(-1)\mathbf{2}+\mathbf{1}] \\ [(-1)\mathbf{3}+\mathbf{2}]}} \left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ \hline 0 & -1 & 1 \\ -1 & 1 & 0 \\ 1 & 0 & 0 \end{array}\right] = \left[\begin{array}{c} \mathsf{I} \\ \hline \mathsf{A}^{-1} \end{array}\right]$$

(L-8) Question 1(a) True

(L-8) Question 1(b) False

(L-8) Question 1(c) True

(L-8) Question 1(d) False

(L-8) Question 2. $\left[\begin{array}{cccc|c} 1 & 3 & 1 & 2 & -1 \\ 2 & 6 & 4 & 8 & -3 \\ 0 & 0 & 2 & 4 & -1 \\ \hline 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{\substack{\tau \\ [(-3)\mathbf{1}+\mathbf{2}] \\ [(-1)\mathbf{1}+\mathbf{3}] \\ [(-2)\mathbf{1}+\mathbf{4}] \\ [(1)\mathbf{1}+\mathbf{5}]}} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 2 & 0 & 2 & 4 & -1 \\ 0 & 0 & 2 & 4 & -1 \\ \hline 1 & -3 & -1 & -2 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{\substack{\tau \\ [(-2)\mathbf{3}+\mathbf{4}] \\ [(2)\mathbf{5}] \\ [(1)\mathbf{3}+\mathbf{5}]}} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ \hline 1 & -3 & -1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & 2 \end{array}\right] \xrightarrow{\substack{\tau \\ [(\frac{1}{2})\mathbf{5}]}} \left[\begin{array}{cccc|} 1 & 0 & 0 & 0 \\ 2 & 0 & 2 & 0 \\ 0 & 0 & 2 & 0 \\ \hline 1 & -3 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 \\ \hline 0 & 0 & 0 & 0 \end{array}\right.$

$$\left\{ \boldsymbol{x} \in \mathbb{R}^4 \;\middle|\; \exists \boldsymbol{p} \in \mathbb{R}^2 \text{ tal que } \boldsymbol{x} = \begin{pmatrix} \frac{1}{2} \\ 0 \\ \frac{1}{2} \\ 0 \end{pmatrix} + \begin{bmatrix} -3 & 0 \\ 1 & 0 \\ 0 & -2 \\ 0 & 1 \end{bmatrix} \boldsymbol{p} \right\}$$

```
A = Matrix([[1,3,1,2],[2,6,4,8],[0,0,2,4]])
b = Vector([1,3,1])
SEL(A,b,1)
```

□

(L-8) Question 3.

```
A = Matrix([ [1,2,-1,-2,1], [1,2,0,0,3], [2,4,1,2,9] ])
b = Vector( [0,-1,-4] )
SEL(A,b,1)
```

$$\left\{ \boldsymbol{x} \in \mathbb{R}^5 \;\middle|\; \exists \boldsymbol{p} \in \mathbb{R}^2 \text{ tal que } \boldsymbol{x} = \begin{pmatrix} 2 \\ 0 \\ 1 \\ 0 \\ -1 \end{pmatrix} + \begin{bmatrix} 0 & -2 \\ 0 & 1 \\ -2 & 0 \\ 1 & 0 \\ 0 & 0 \end{bmatrix} \boldsymbol{p} \right\}$$

□

(L-8) Question 4. Para expresar la solución completa del sistema necesitamos una solución particular a la que sumar cualquier combinación de los vectores del espacio nulo (soluciones del sistema homogéneo).

Para obtener una solución particular podríamos aplicar la eliminación gaussiana pero en este caso una solución inmediata es asignar el valor uno a x_3 y x_4; y cero a las demás; es decir, sumar las columnas 3 y 4 de la matriz de coeficientes. Por tanto una solución particular inmediata es

$$\boldsymbol{x}_p = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}$$

Para calcular el espacio nulo aplicaremos el método de eliminación a la matriz de coeficientes del sistema de ecuaciones:

$$\left[\begin{array}{ccccc} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ \hline 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow[{[(-1)\mathbf{1}+\mathbf{5}]}]{\substack{\boldsymbol{\tau} \\ [(-1)\mathbf{1}+\mathbf{3}]}} \left[\begin{array}{ccccc} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 & 0 \\ \hline 1 & 0 & -1 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow[{[(-1)\mathbf{3}+\mathbf{4}]}]{\substack{\boldsymbol{\tau} \\ [(-1)\mathbf{2}+\mathbf{4}]}} \left[\begin{array}{ccccc} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 \\ \hline 1 & 0 & -1 & 1 & -1 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

Así pues, la solución completa al sistema es el conjunto de vectores

$$\left\{ \boldsymbol{x} \in \mathbb{R}^5 \;\middle|\; \exists c \in \mathbb{R} \text{ such that } \boldsymbol{x} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} + c \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \\ 0 \end{pmatrix} \right\}$$

□

(L-8) Question 5(a)

$$\left[\begin{array}{c|c} \mathsf{A} & \textbf{-}\boldsymbol{b} \\ \hline \mathsf{I} & \mathbf{0} \\ \hline \mathbf{0} & 1 \end{array}\right] = \left[\begin{array}{cccc|c} 0 & 1 & 0 & 3 & -b_1 \\ 0 & 2 & 0 & 6 & -b_2 \\ \hline 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{\substack{\boldsymbol{\tau} \\ [(-3)\mathbf{2}+\mathbf{4}]}} \left[\begin{array}{cccc|c} 0 & 1 & 0 & 0 & -b_1 \\ 0 & 2 & 0 & 0 & -b_2 \\ \hline 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -3 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{\substack{\boldsymbol{\tau} \\ [(b_1)\mathbf{2}+\mathbf{5}]}} \left[\begin{array}{cccc|c} 0 & 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 2b_1 - b_2 \\ \hline 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -3 & b_1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

The column reduced echelon form is

$$\mathsf{R} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \end{bmatrix}.$$

(L-8) Question 5(b) the variables x_1, x_3 and x_4 are free.

(L-8) Question 5(c)

$$\boldsymbol{x}_a = \begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \quad \boldsymbol{x}_b = \begin{pmatrix}0\\0\\1\\0\end{pmatrix}, \quad \boldsymbol{x}_c = \begin{pmatrix}0\\-3\\0\\1\end{pmatrix}$$

(L-8) Question 5(d) The linear system is consistent when $b_2 = 2b_1$. In that case by gaussian elimination we get

$$\left[\begin{array}{c|c} \mathbf{R} & \mathbf{0} \\ \hline \mathbf{E} & \boldsymbol{x}_p \\ \hline \mathbf{0} & 1 \end{array}\right].$$

(L-8) Question 5(e) In that case ($b_2 = 2b_1$) a particular solution is

$$\boldsymbol{x}_p = \begin{pmatrix}0\\b_1\\0\\0\end{pmatrix}$$

And then, the complete solution to the system is

$$\left\{ \boldsymbol{x} \in \mathbb{R}^4 \;\middle|\; \exists \begin{pmatrix}a\\c\\d\end{pmatrix} \text{ such that } \boldsymbol{x} = \begin{pmatrix}0\\b_1\\0\\0\end{pmatrix} + \begin{bmatrix}1&0&0\\0&0&-3\\0&1&0\\0&0&1\end{bmatrix}\begin{pmatrix}a\\c\\d\end{pmatrix} = \begin{pmatrix}a\\b_1-3d\\c\\d\end{pmatrix} \right\}$$

```
b1,b2 = sympy.symbols('b1 b2')
A  = Matrix([[0,1,0,3],[0,2,0,6]])
b  = Vector([b1,b2])
SEL(A, b, 1)
```

(L-8) Question 6.

$$\left[\begin{array}{c|c} \mathbf{A} & \text{-}\boldsymbol{b} \\ \hline \mathbf{I} & \mathbf{0} \\ \hline \mathbf{0} & 1 \end{array}\right] = \left[\begin{array}{cc|c} 0&0&\text{-}b_1\\ 1&2&\text{-}b_2\\ 0&0&\text{-}b_3\\ 3&6&\text{-}b_4\\ \hline 1&0&0\\ 0&1&0\\ \hline 0&0&1 \end{array}\right] \xrightarrow{\overset{\tau}{[(-2)\mathbf{1}+\mathbf{2}]}} \left[\begin{array}{cc|c} 0&0&\text{-}b_1\\ 1&0&\text{-}b_2\\ 0&0&\text{-}b_3\\ 3&0&\text{-}b_4\\ \hline 1&-2&0\\ 0&1&0\\ \hline 0&0&1 \end{array}\right] \xrightarrow{\overset{\tau}{[(b_2)\mathbf{1}+\mathbf{3}]}} \left[\begin{array}{cc|c} 0&0&\text{-}b_1\\ 1&0&0\\ 0&0&\text{-}b_3\\ 3&0&3b_2\text{-}b_4\\ \hline 1&-2&b_2\\ 0&1&0\\ \hline 0&0&1 \end{array}\right]$$

Therefore, the reduced row echelon form is

$$\mathbf{R} = \begin{bmatrix}1&0\\0&0\\3&0\\0&0\end{bmatrix};$$

and x_2 is a free variable. The special solution is $\boldsymbol{x}_a = \begin{pmatrix}-2\\1\end{pmatrix}$. The system is consistent when $b_1 = 0$, $b_3 = 0$ and

$b_4 = 3b_2$. In that case by gaussian elimination we get

$$\left[\begin{array}{c|c} \mathbf{R} & \mathbf{0} \\ \hline \mathbf{E} & \boldsymbol{x}_p \\ \hline \mathbf{0} & 1 \end{array}\right],$$

and then a particular solution is $\boldsymbol{x}_p = \begin{pmatrix} b_2 \\ 0 \end{pmatrix}$. Then, the complete solution to the system is

$$\left\{ \boldsymbol{x} \in \mathbb{R}^2 \;\middle|\; \exists a \in \mathbb{R} \text{ such that } \boldsymbol{x} = \begin{pmatrix} b_2 \\ 0 \end{pmatrix} + a \begin{pmatrix} -2 \\ 1 \end{pmatrix} \right\}$$

□

(L-8) Question 7.

$$\left[\begin{array}{c|c} \mathbf{A} & -\boldsymbol{b} \\ \hline \mathbf{I} & \mathbf{0} \\ \hline \mathbf{0} & 1 \end{array}\right] = \left[\begin{array}{cc|c} 1 & 0 & -b_1 \\ 0 & 1 & -b_2 \\ 2 & 3 & -b_3 \\ \hline 1 & 0 & 0 \\ 0 & 1 & 0 \\ \hline 0 & 0 & 1 \end{array}\right] \xrightarrow[{[(b_2)\mathbf{2+3}]}]{\substack{\boldsymbol{\tau} \\ [(b_1)\mathbf{1+3}]}} \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 3 & 2b_1 + 3b_2 - b_3 \\ \hline 1 & 0 & b_1 \\ 0 & 1 & b_2 \\ \hline 0 & 0 & 1 \end{array}\right]$$

then, the condition is $b_3 - 2b_1 - 3b_2 = 0$; or

$$\mathcal{C}(\mathbf{A}) = \left\{ \boldsymbol{b} \in \mathbb{R}^3 \;\middle|\; b_3 - 2b_1 - 3b_2 = 0 \right\} = \left\{ \boldsymbol{b} \in \mathbb{R}^3 \;\middle|\; \begin{bmatrix} -2 & -3 & 1 \end{bmatrix} \boldsymbol{b} = \mathbf{0} \right\}.$$

The rank of $\mathbf{A}$ is 2. An attainable right-hand side $\boldsymbol{b}$ is

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 2 & 3 \end{bmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 5 \end{pmatrix}.$$

The null space only has the zero vector $\mathbf{0} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ (no free columns).

□

(L-8) Question 8. We must solve

$$c \begin{pmatrix} 10 \\ 30 \\ 12 \end{pmatrix} + t \begin{pmatrix} 20 \\ 75 \\ 36 \end{pmatrix} + p \begin{pmatrix} 40 \\ 135 \\ 64 \end{pmatrix} = \begin{bmatrix} 10 & 20 & 40 \\ 30 & 75 & 135 \\ 12 & 36 & 64 \end{bmatrix} \begin{pmatrix} c \\ t \\ p \end{pmatrix} = \begin{pmatrix} 760 \\ 2595 \\ 1224 \end{pmatrix} \implies \boldsymbol{x} = \begin{pmatrix} 4 \\ 6 \\ 15 \end{pmatrix};$$

Then they paint 4 cars, 6 trains and 15 planes each week.

```
A = Matrix([ [10, 20, 40], [30, 75, 135], [12, 36, 64] ])
b = Vector( [760, 2595, 1224] )
SEL(A,b,1)
```

□

(L-8) Question 9(a)

$$\left[\begin{array}{c|c} \mathbf{A} & -\boldsymbol{b} \\ \hline \mathbf{I} & \mathbf{0} \\ \hline \mathbf{0} & 1 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 0 & 4 & -6 \\ 2 & 1 & 10 & -14 \\ 3 & 1 & c & -20 \\ \hline 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow[{[(6)\mathbf{1+4}]}]{\substack{\boldsymbol{\tau} \\ [(-4)\mathbf{1+3}]}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 2 & 1 & 2 & -2 \\ 3 & 1 & c-12 & -2 \\ \hline 1 & 0 & -4 & 6 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow[{[(2)\mathbf{2+4}]}]{\substack{\boldsymbol{\tau} \\ [(-4)\mathbf{2+1}] \\ [(-2)\mathbf{2+3}]}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & c-14 & 0 \\ \hline 1 & 0 & -4 & 6 \\ -2 & 1 & -2 & 2 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right] = \left[\begin{array}{c|c} \mathbf{L} & \mathbf{0} \\ \hline \mathbf{E} & \boldsymbol{x}_p \\ \hline \mathbf{0} & 1 \end{array}\right]$$

Vamos a operar con la matriz ampliada $\left[\mathbf{A}| - \boldsymbol{b}\right]$, para que los cálculos nos sirvan para el apartado siguiente. Por tanto, $\mathbf{A}$ tendrá menos de 3 pivotes (y por tanto no será invertible) si $c = 14$.

□

(L-8) Question 9(b) Cuando $c = 14$, las dos primeras variables x_1 y x_2 son pivote, y la tercera es libre. Puesto que solo hay una columna libre, sólo necesitamos una solución del sistema homogéneo para obtener una base del espacio nulo $\mathcal{N}\left(\mathbf{A}\right)$. Así pues

$$\text{Sol.} = \left\{ \boldsymbol{x} \in \mathbb{R}^3 \;\middle|\; \exists a \in \mathbb{R} \text{ such that } \boldsymbol{x} = \begin{pmatrix} 6 \\ 2 \\ 0 \end{pmatrix} + a \begin{pmatrix} -4 \\ -2 \\ 1 \end{pmatrix} \right\}$$

□

(L-8) Question 9(c) Este sistema (con $c = 14$) tiene un espacio nulo $\mathcal{N}\left(\mathbf{A}\right)$ de dimensión uno (infinitas soluciones). Así pues, sus filas representan tres planos en $\mathbb{R}^3$ que se cortan en una sola recta. Visto por columnas, y puesto que sólo dos de ellas son pivote —y la tercera es libre— el sistema tiene infinitas soluciones, ya que hay infinitas combinaciones de las tres columnas que general el vector del lado derecho $\boldsymbol{b}$.

□

(L-8) Question 10. Let's use the first two vectors as columns one and two of $\mathbf{A}$

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & a_{13} \\ 1 & 3 & a_{23} \\ 5 & 1 & a_{33} \end{bmatrix}$$

We know

$$\begin{bmatrix} 1 & 0 & a_{13} \\ 1 & 3 & a_{23} \\ 5 & 1 & a_{33} \end{bmatrix} \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} = \mathbf{0}; \Rightarrow \begin{pmatrix} 1 \\ 1 \\ 5 \end{pmatrix} + \begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix} + 2 \begin{pmatrix} a_{13} \\ a_{23} \\ a_{33} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

therefore

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & -1/2 \\ 1 & 3 & -2 \\ 5 & 1 & -3 \end{bmatrix}$$

□

(L-8) Question 11(a) It has a solution for any $\boldsymbol{b}$ (three pivots in $\mathbb{R}^3$).

□

(L-8) Question 11(b) It has solution only if $b_3 = 0$.

□

(L-8) Question 12.

$$\left[\begin{array}{cccc|c} 1 & 2 & 0 & 3 & -b_1 \\ 2 & 4 & 0 & 7 & -b_2 \\ \hline 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow[{[(-3)\mathbf{1}+\mathbf{4}]}]{\substack{\tau \\ [(-2)\mathbf{1}+\mathbf{2}]}} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & -b_1 \\ 2 & 0 & 0 & 1 & -b_2 \\ \hline 1 & -2 & 0 & -3 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{\substack{\tau \\ [(-2)\mathbf{4}+\mathbf{1}]}}$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & -b_1 \\ 0 & 0 & 0 & 1 & -b_2 \\ \hline 7 & -2 & 0 & -3 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ -2 & 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow[{[(b_2)\mathbf{2}+\mathbf{5}]}]{\substack{\tau \\ [(b_1)\mathbf{1}+\mathbf{5}]}} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ \hline 7 & -2 & 0 & -3 & -3b_2 + 7b_1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ -2 & 0 & 0 & 1 & b_2 - 2b_1 \\ \hline 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

Since the rank($\mathbf{A}$) is 2, the system is solvable for any $\boldsymbol{b}$.

The general solution is the set of vectors

$$\left\{ \boldsymbol{x} \in \mathbb{R}^4 \;\middle|\; \exists\, a, b \in \mathbb{R} \text{ such that } \boldsymbol{x} = \begin{pmatrix} -3b_2 + 7b_1 \\ 0 \\ 0 \\ b_2 - 2b_1 \end{pmatrix} + c \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + d \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right\}$$

□

(L-8) Question 13(a) $\mathbf{B}$ es de orden 3 por 4; por tanto los vectores de $\mathcal{N}\left(\mathbf{B}\right)$ pertenecen a $\mathbb{R}^4$.

Por otra parte, puesto que la segunda matriz del producto (llamemosla $\mathbf{E}$) es de rango completo, sabemos que es producto de matrices elementales y por tanto invertible.

Si llamamos a la matriz del producto del enunciado $\mathbf{L}$, tenemos que

$$\mathbf{B} = \mathbf{L}\mathbf{E} \Rightarrow \mathbf{B}\mathbf{E}^{-1} = \mathbf{L}$$

Por tanto, las columnas nulas de $\mathbf{L}$ son combinaciones lineales de las columnas de $\mathbf{B}$, y las columnas de $\mathbf{E}^{-1}$ nos indican qué combinaciones son. Como $\mathbf{L}$ tiene dos pivotes, el rango de $\mathbf{B}$ es dos.

Para encontrar las soluciones del espacio nulo basta con invertir $\mathbf{E}$ y tomar sus dos últimas columnas (las que transforman las columnas de $\mathbf{B}$ en columnas de ceros. Así pues,

$$\left[\begin{array}{c} \mathbf{E} \\ \hline \mathbf{I} \end{array}\right] = \left[\begin{array}{cccc} 1 & 1 & -0 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & -0 \\ 0 & 0 & 0 & 1 \\ \hline 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \hline 1 & -1 & 1 & -2 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right] = \left[\begin{array}{c} \mathbf{I} \\ \hline \mathbf{E}^{-1} \end{array}\right]$$

El espacio nulo es

$$\mathcal{N}\left(\mathbf{B}\right) = \left\{ \boldsymbol{x} \in \mathbb{R}^4 \;\middle|\; \exists\, a, b \in \mathbb{R} \text{ such that } \boldsymbol{x} = a \begin{pmatrix} -1 \\ -1 \\ 1 \\ 0 \end{pmatrix} + b \begin{pmatrix} -2 \\ 1 \\ 0 \\ 1 \end{pmatrix} \right\}$$

□

(L-8) Question 13(b) A la vista de la primera columna de las matrices que intervienen en el producto, sabemos que la primera columna de $\mathbf{B}$ es precisamente $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$. Por tanto, una solución particular es

$$\boldsymbol{x}_p = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Y dado que conocemos el espacio nulo de $\mathbf{B}$, la solución completa es

$$\left\{ \boldsymbol{x} \in \mathbb{R}^4 \;\middle|\; \exists\, a, b \in \mathbb{R} \text{ such that } \boldsymbol{x} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + a \begin{pmatrix} -1 \\ -1 \\ 1 \\ 0 \end{pmatrix} + b \begin{pmatrix} -2 \\ 1 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

□

(L-8) Question 14(a)

$$\left[\begin{array}{rrr|c} 1 & 4 & 2 & \text{-}b_1 \\ 2 & 8 & 4 & \text{-}b_2 \\ -1 & -4 & -2 & \text{-}b_3 \\ \hline 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow[\substack{[(-4)\mathbf{1}+\mathbf{2}] \\ [(-2)\mathbf{1}+\mathbf{3}]}]{\tau} \left[\begin{array}{rrr|c} 1 & 0 & 0 & \text{-}b_1 \\ 2 & 0 & 0 & \text{-}b_2 \\ -1 & 0 & 0 & \text{-}b_3 \\ \hline 1 & -4 & -2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow[{[(b_1)\mathbf{1}+\mathbf{4}]}]{\tau} \left[\begin{array}{rrr|c} 1 & 0 & 0 & 0 \\ 2 & 0 & 0 & 2b_1-b_2 \\ -1 & 0 & -0 & -b_1-b_3 \\ \hline 1 & -4 & -2 & b_1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right]$$

Solution only if $b_2 = 2b_1$ and $b_3 = -b_1$. Since the rank is one, the column space is a line. $\mathbb{R}^3$

□

(L-8) Question 14(b)

$$\left[\begin{array}{rr|c} 1 & 4 & \text{-}b_1 \\ 2 & 9 & \text{-}b_2 \\ -1 & -4 & \text{-}b_3 \\ \hline 1 & 0 & 0 \\ 0 & 1 & 0 \\ \hline 0 & 0 & 1 \end{array}\right] \xrightarrow[\substack{[(-4)\mathbf{1}+\mathbf{2}] \\ [(-2)\mathbf{2}+\mathbf{1}]}]{\tau} \left[\begin{array}{rr|c} 1 & 0 & \text{-}b_1 \\ 0 & 1 & \text{-}b_2 \\ -1 & 0 & \text{-}b_3 \\ \hline 9 & -4 & 0 \\ -2 & 1 & 0 \\ \hline 0 & 0 & 1 \end{array}\right] \xrightarrow[{[(b_1)\mathbf{1}+\mathbf{3}]}]{\tau} \left[\begin{array}{rr|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & -b_1-b_3 \\ \hline 9 & -4 & 9b_1-4b_2 \\ -2 & 1 & b_2 \\ \hline 0 & 0 & 1 \end{array}\right]$$

Solution only if $b_3 = -b_1$. Since the rank is two, the column space is a plane.

□

(L-8) Question 15. Assuming that $\boldsymbol{b}$ equals

$$\boldsymbol{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix} \text{ then the matrix } \mathsf{A} \text{ will have the form } \quad \mathsf{A} = \begin{bmatrix} b_1 & 0 & 0 \\ b_2 & 0 & 0 \\ \vdots & \vdots & \vdots \\ b_n & 0 & 0 \end{bmatrix} = \begin{bmatrix} \boldsymbol{b} & \mathbf{0} & \mathbf{0} \end{bmatrix}.$$

□

(L-8) Question 16(a) The linear system is

$$\begin{bmatrix} 1 & 6 & 1 \\ 0 & 2 & 3 \\ 1 & 5 & 0 \end{bmatrix} \begin{pmatrix} r \\ b \\ g \end{pmatrix} = \begin{pmatrix} 44 \\ 24 \\ 33 \end{pmatrix}, \quad \text{cuya solución es :} \quad g = 2;\ b = 9;\ r = -12.$$

```
A = Matrix([ [1,6,1], [0,2,3], [1,5,0] ])
b = Vector( [44,24,33] )
SEL(A,b,1)
```

□

(L-8) Question 16(b) Old Economy Answer: The store is paying people 12 per gallon to take its red paint. Something doesn't make sense.

□

(L-8) Question 16(c) We can figure out who was undercharged by trial and error. We have to add 4 euros to the right hand side of one of the three equations above, and then solve it again. After some computation, we find that the only way we can get a nonzero solution is if we add 4 euros to the second equation. Therefore Shai was overcharged.

□

(L-8) Question 16(d) Se pueden resolver los tres sistemas del apartado anterior de una sola vez siguiendo las instrucciones. Partimos de la matriz ampliada (y aprovechamos los pasos dados en el primer apartado)

$$\left[\begin{array}{ccc|ccc} 1 & 6 & 1 & -48 & -44 & -44 \\ 0 & 2 & 3 & -24 & -28 & -24 \\ 1 & 5 & 0 & -33 & -33 & -37 \\ \hline 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -48 & -44 & -44 \\ 0 & 1 & 0 & -24 & -28 & -24 \\ 0 & 0 & 1 & -33 & -33 & -37 \\ \hline -15 & 5 & 16 & 0 & 0 & 0 \\ 3 & -1 & -3 & 0 & 0 & 0 \\ -2 & 1 & 2 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow[{[(37)\mathbf{3+6}]}]{\substack{\tau \\ [(48)\mathbf{1+4}] \\ [(24)\mathbf{2+4}] \\ [(33)\mathbf{3+4}] \\ [(44)\mathbf{1+5}] \\ [(28)\mathbf{2+5}] \\ [(33)\mathbf{3+5}] \\ [(44)\mathbf{1+6}] \\ [(24)\mathbf{2+6}]}} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \hline -15 & 5 & 16 & -72 & 8 & 52 \\ 3 & -1 & -3 & 21 & 5 & -3 \\ -2 & 1 & 2 & -6 & 6 & 10 \\ \hline 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

como se puede ver, sólo al alterar el importe de la factura de Belén (de 24 a 28 euros), se encuentran tres precios positivos

$$r = 8; \quad a = 5; \qquad v = 6$$

Moraleja: con el método de eliminación Gauss-Jordan se pueden resolver varios sistemas de ecuaciones a la vez; siempre y cuando compartan la misma matriz de coeficientes $\mathbf{A}$.

□

(L-8) Question 17(a) Si $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$ tiene solución, ello significa que existen $x_1, x_2, \ldots, x_n$ tales que

$$x_1\mathbf{A}_{|1} + x_2\mathbf{A}_{|2} + \cdots + x_n\mathbf{A}_{|n} = \boldsymbol{b}$$

pero entonces, $\boldsymbol{b}$ es una combinación lineal de las columnas de $\mathbf{A}$, y por lo tanto $\boldsymbol{b} \in \mathcal{C}\left(\mathbf{A}\right)$.

□

(L-8) Question 17(b) Puesto que $\mathbf{A}\boldsymbol{x}_0 = \boldsymbol{b}$ y que $\mathbf{A}\boldsymbol{z} = \mathbf{0}$ sumando ambas ecuaciones tenemos

$$\begin{aligned} &\mathbf{A}\boldsymbol{x}_0 + \mathbf{A}\boldsymbol{z} = \boldsymbol{b} + \mathbf{0} \\ &\mathbf{A}(\boldsymbol{x}_0 + \boldsymbol{z}) = \boldsymbol{b} && \text{sacando } \mathbf{A} \text{ como factor común.} \end{aligned}$$

Por tanto el vector $\boldsymbol{x}_0 + \boldsymbol{z}$ también es solución del sistema.

□

(L-8) Question 17(c) Dado el resultado del apartado anterior, basta con demostrar que si hay dependencia lineal entre las columnas de $\mathbf{A}$, el sistema homogéneo tiene soluciones distintas de la trivial ($\boldsymbol{x} = \mathbf{0}$).

Si hay dependencia lineal entre las columnas significa que al menos una de ellas se puede expresar como combinación lineal de las demás. Supongamos sin pérdida de generalidad que es la primera; entonces

$$\mathbf{A}_{|1} = z_2\mathbf{A}_{|2} + z_3\mathbf{A}_{|3} + \cdots + z_m\mathbf{A}_{|m}$$

pasando la expresión de la derecha al lado izquierdo de la igualdad tenemos:

$$\mathbf{A}_{|1} - z_2\mathbf{A}_{|2} - z_3\mathbf{A}_{|3} - \cdots - z_m\mathbf{A}_{|m} = \mathbf{0}$$

$$\begin{bmatrix}\mathbf{A}_{|1} & \mathbf{A}_{|2} & \cdots & \mathbf{A}_{|m}\end{bmatrix} \begin{pmatrix} 1 \\ -z_2 \\ \vdots \\ -z_m \end{pmatrix} = \mathbf{0};$$

por tanto el vector $\begin{pmatrix}1 & -z_2 & \cdots & -z_m\end{pmatrix}$ y cualquier múltiplo de este son solución al sistema de ecuaciones homogéneo (pertenecen a $\mathcal{N}\left(\mathbf{A}\right)$). Este resultado unido al anterior demuestran que el sistema tiene más de una solución (de hecho tiene infinitas, ya que hay infinitos vectores en el espacio nulo $\mathcal{N}\left(\mathbf{A}\right)$).

□

(L-8) Question 18. Note that the first column plus three times the third gives $\boldsymbol{b}$. Let's see how Gaussian elimination finds the solution:

$$\left[\begin{array}{ccc|c} 3 & 1 & 1 & -6 \\ 1 & -1 & -1 & 2 \\ 0 & 4 & 1 & -3 \\ \hline 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow[{[(6)\mathbf{3}+\mathbf{4}]}]{\substack{\tau \\ [(-1)\mathbf{3}+\mathbf{2}] \\ [(-3)\mathbf{3}+\mathbf{1}]}} \left[\begin{array}{ccc|c} 0 & 0 & 1 & 0 \\ 4 & 0 & -1 & -4 \\ -3 & 3 & 1 & 3 \\ \hline 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -3 & -1 & 1 & 6 \\ \hline 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{\substack{\tau \\ [(1)\mathbf{1}+\mathbf{4}]}} \left[\begin{array}{ccc|c} 0 & 0 & 1 & 0 \\ 4 & 0 & -1 & 0 \\ -3 & 3 & 1 & 0 \\ \hline 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ -3 & -1 & 1 & 3 \\ \hline 0 & 0 & 0 & 1 \end{array}\right]$$

$$\boldsymbol{x}_p = \begin{pmatrix} 1 & 0 & 3 \end{pmatrix}.$$

□

(L-8) Question 19(a)

$$\begin{cases} x = 2y \\ x + y = 39 \end{cases} \qquad \text{therefore} \qquad \begin{bmatrix} 1 & -2 \\ 1 & 1 \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 39 \end{pmatrix}$$

Using Gaussian elimination

$$\left[\begin{array}{cc|c} 1 & -2 & 0 \\ 1 & 1 & -39 \\ \hline 1 & 0 & 0 \\ 0 & 1 & 0 \\ \hline 0 & 0 & 1 \end{array}\right] \xrightarrow{\substack{\tau \\ [(2)\mathbf{1}+\mathbf{2}]}} \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 1 & 3 & -39 \\ \hline 1 & 2 & 0 \\ 0 & 1 & 0 \\ \hline 0 & 0 & 1 \end{array}\right] \xrightarrow{\substack{\tau \\ [(13)\mathbf{2}+\mathbf{3}]}} \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 1 & 3 & 0 \\ \hline 1 & 2 & 26 \\ 0 & 1 & 13 \\ \hline 0 & 0 & 1 \end{array}\right]$$

Hence $x = 26$ and $y = 13$.

□

(L-8) Question 19(b) Since both points lie on the line, we can substitute x and y:

$$\begin{cases} 5 = 2m + c \\ 7 = 3m + c \end{cases} \qquad \text{therefore} \qquad \begin{bmatrix} 2 & 1 \\ 3 & 1 \end{bmatrix} \begin{pmatrix} m \\ c \end{pmatrix} = \begin{pmatrix} 5 \\ 7 \end{pmatrix}$$

By elimination

$$\left[\begin{array}{cc|c} 2 & 1 & -5 \\ 3 & 1 & -7 \\ \hline 1 & 0 & 0 \\ 0 & 1 & 0 \\ \hline 0 & 0 & 1 \end{array}\right] \xrightarrow{\substack{\tau \\ [(-2)\mathbf{2}+\mathbf{1}] \\ [(5)\mathbf{2}+\mathbf{3}]}} \left[\begin{array}{cc|c} 0 & 1 & 0 \\ 1 & 1 & -2 \\ \hline 1 & 0 & 0 \\ -2 & 1 & 5 \\ \hline 0 & 0 & 1 \end{array}\right] \xrightarrow{\substack{\tau \\ [(2)\mathbf{1}+\mathbf{3}]}} \left[\begin{array}{cc|c} 0 & 1 & 0 \\ 1 & 1 & 0 \\ \hline 1 & 0 & 2 \\ -2 & 1 & 1 \\ \hline 0 & 0 & 1 \end{array}\right]$$

Hence, $c = 1$ and $m = 2$.

□

(L-8) Question 20. The linear system is

$$\begin{aligned} a + b + c &= 4 \\ a + 2b + 4c &= 8 \\ a + 3b + 9c &= 14 \end{aligned}$$

by column reduction we get:

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & -4 \\ 1 & 2 & 4 & -8 \\ 1 & 3 & 9 & -14 \\ \hline 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow[{[(4)\mathbf{1}+\mathbf{4}]}]{\substack{\tau \\ [(-1)\mathbf{1}+\mathbf{2}] \\ [(-1)\mathbf{1}+\mathbf{3}]}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 1 & 1 & 3 & -4 \\ 1 & 2 & 8 & -10 \\ \hline 1 & -1 & -1 & 4 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow[{[(4)\mathbf{2}+\mathbf{4}]}]{\substack{\tau \\ [(-3)\mathbf{2}+\mathbf{3}]}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 2 & 2 & -2 \\ \hline 1 & -1 & 2 & 0 \\ 0 & 1 & -3 & 4 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{\substack{\tau \\ [(1)\mathbf{3}+\mathbf{4}]}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 2 & 2 & 0 \\ \hline 1 & -1 & 2 & 2 \\ 0 & 1 & -3 & 1 \\ 0 & 0 & 1 & 1 \\ \hline 0 & 0 & 0 & 1 \end{array}\right];$$

so $a = 2$, $b = 1$ and $c = 1$. Therefore, the parabola is $y = 2 + x + x^2$.

□

(L-8) Question 21. Assume $c = 0$, so $\boldsymbol{b} = \big(2, \quad 1, \quad (0 + c),\big)$.

$$\left[\begin{array}{c|c} \mathsf{A} & \textbf{-}\boldsymbol{b} \\ \hline \mathsf{I} & \mathbf{0} \\ \hline \mathbf{0} & 1 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & -2 \\ 1 & 2 & 3 & -1 \\ 0 & 1 & 2 & 0-c \\ \hline 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{\substack{\tau \\ [(-1)\mathbf{1}+\mathbf{2}] \\ [(-1)\mathbf{1}+\mathbf{3}] \\ [(2)\mathbf{1}+\mathbf{4}]}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 1 & 1 & 2 & 1 \\ 0 & 1 & 2 & 0-c \\ \hline 1 & -1 & -1 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{\substack{\tau \\ [(-2)\mathbf{2}+\mathbf{3}] \\ [(-1)\mathbf{2}+\mathbf{4}]}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1-c \\ \hline 1 & -1 & 1 & 3 \\ 0 & 1 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right] = \left[\begin{array}{c|c} \mathsf{L} & -\boldsymbol{c} \\ \hline \mathsf{E} & \boldsymbol{x} \\ \hline \mathbf{0} & 1 \end{array}\right]$$

is not possible to eliminate the third component of $\boldsymbol{c}$ since there are only two pivots. We should replace the zero by a -1 in $\boldsymbol{b}$.

Then a soluton is $\boldsymbol{x}_p = \begin{pmatrix} 3 \\ -1 \\ 0 \end{pmatrix}$; but we have many more, for example: $\boldsymbol{x} = \begin{pmatrix} 3 \\ -1 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$.

□

(L-8) Question 22. Applying gaussian elimination on the coefficient matrix we get

$$\mathsf{A} = \begin{bmatrix} 2 & 4 \\ b & 8 \end{bmatrix} \xrightarrow{\substack{\tau \\ [(-2)\mathbf{1}+\mathbf{2}]}} \begin{bmatrix} 2 & 0 \\ b & 8-2b \end{bmatrix} = \mathsf{L}$$

The system is singular if $b = 4$, because $4x + 8y$ is 2 times $2x + 4y$.

$$\begin{bmatrix} 2 & 4 \\ 4 & 8 \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 16 \\ g \end{pmatrix}$$

that is

$$x \begin{pmatrix} 2 \\ 4 \end{pmatrix} + y \begin{pmatrix} 4 \\ 8 \end{pmatrix} = \begin{pmatrix} 16 \\ g \end{pmatrix}$$

The system is asking us to find a linear combination of columns of A that equals $\begin{pmatrix} 16 \\ g \end{pmatrix}$.

But in this case, the second column is twice the first (this columns are align); hence, the set of all their linear combinations are a line (all of them lie on the same line), so the system has solution only if the right hand side vector lie on that line (only if it is a multimple of the columns of A. That happens when $g = 32$:

$$\begin{bmatrix} 2 & 4 \\ 4 & 8 \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 16 \\ 32 \end{pmatrix}$$

Two possible solutions are

$$\begin{cases} x = 8 \\ y = 0 \end{cases} \qquad \text{or} \qquad \begin{cases} x = 0 \\ y = 4. \end{cases}$$

In fact there are infinite solutions.

□

(L-8) Question 23. By back substitution

$$\begin{aligned} w &= b_3 \\ v &= b_2 - w = b_2 - b_3 \\ u &= b_1 + v - w = b_1 + (b_2 - b_3) - b_3 = b_1 + b_2 - 2b_3 \end{aligned}$$

Then b_3 times the third column, plus $b_2 - b_3$ times the second one, plus $b_1 + b_2 - 2b_3$ times the first one equals $\boldsymbol{b}$.

Let's check the solution:

$$\begin{bmatrix}1&1&1\\&1&1\\&&1\end{bmatrix}\begin{pmatrix}b_1+b_2-2b_3\\b_2-b_3\\b_3\end{pmatrix}=(b_1+b_2-2b_3)\begin{pmatrix}1\\0\\0\end{pmatrix}+(b_2-b_3)\begin{pmatrix}1\\1\\0\end{pmatrix}+b_3\begin{pmatrix}1\\1\\1\end{pmatrix}=\begin{pmatrix}(b_1+b_2-2b_3)-(b_2-b_3)+b3\\(b_2-b_3)+b_3\\b_3\end{pmatrix}=\begin{pmatrix}b_1\\b_2\\b_3\end{pmatrix}$$

□

(L-8) Question 24(a) Since there is only one solution in $\mathbb{R}^2$, both columns of $\mathbf{A}$ must be linearly independent (rank 2). We also know the second column is equal to the right hand side vector. Then, any matrix as

$$\mathbf{A}=\begin{bmatrix}a&1\\b&2\\c&3\end{bmatrix}$$

where the vectors $\begin{pmatrix}a\\b\\c\end{pmatrix}$ and $\begin{pmatrix}1\\2\\3\end{pmatrix}$ are linearly independent is OK; for example

$$\mathbf{A}=\begin{bmatrix}1&1\\0&2\\0&3\end{bmatrix}$$

□

(L-8) Question 24(b) There is not such a matrix $\mathbf{B}$: the right hand side vector of $\mathbb{R}^2$ tells us that we have two equations; but the vector $\boldsymbol{x}$ of $\mathbb{R}^3$ says "there are three unknowns"... in that case there is no solution at all, or there are an infinite number of solutions... but the exercise claims that there is only one!

□

(L-8) Question 25. On the one hand, the right hand side vector of $\mathbb{R}^2$ indicates there are only two equations ($\mathbf{A}$ has two rows); the vector of unknowns $\boldsymbol{x}$ is also of order two: two unknows ($\mathbf{A}$ has two columns); and the particular solution tells us that the first column is $\begin{pmatrix}1\\3\end{pmatrix}$.

On the other hand, the solution to the homogeneus system is any multiple of the second column; therefore, the second column must be the zero vector. Then

$$\mathbf{A}=\begin{bmatrix}1&0\\3&0\end{bmatrix}.$$

□

(L-8) Question 26. free
is not
infinite

□

(L-8) Question 27(a) The system could have no solution; for example

$$\begin{bmatrix}1&1&0&0\\2&2&0&0\\0&0&0&0\end{bmatrix}\begin{pmatrix}x_1\\x_2\\x_3\\x_4\end{pmatrix}=\begin{pmatrix}1\\1\\1\end{pmatrix}$$

Nótese que el rango de la matriz de coeficientes $\mathbf{A}$ es uno, pero el rango de la matriz ampliada $[\mathbf{A}|\boldsymbol{b}]$ es dos.

□

(L-8) Question 27(b) Since there are more equations than unknowns, if the system has a solution, that solution is not unique.

□

(L-8) Question 27(c) The rigth hand side vector $\boldsymbol{b}$ must be a lienar combination of the columns of $\mathbf{A}$; in other words, the matrices $\mathbf{A}$ and $[\mathbf{A}|\boldsymbol{b}]$ must have the same rank.

□

(L-8) Question 27(d) Since $\boldsymbol{b}$ belongs to $\mathbb{R}^3$, the rank of $\mathbf{A}$ must be 3. □

(L-8) Question 28(a) The rank is 4 (there are 4 pivots in the reduced echelon form of $\mathbf{A}$).

$$\left\{ \boldsymbol{x} \in \mathbb{R}^7 \;\middle|\; \exists\, a, b, c \in \mathbb{R} \text{ such that } \boldsymbol{x} = a \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + b \begin{pmatrix} -2 \\ 0 \\ -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + c \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right\}$$

(parametric equation) □

(L-8) Question 28(b) if $a = x_2,\ b = x_4$ and $c = x_6$:

$$\left\{ \boldsymbol{x} \in \mathbb{R}^7 \;\middle|\; \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{pmatrix} = x_2 \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -2 \\ 0 \\ -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_6 \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} x_2 - 2x_4 + x_6 \\ x_2 \\ -x_4 - x_6 \\ x_4 \\ 0 \\ x_6 \\ 0 \end{pmatrix} \right\}$$

so, it is the subset of vectors in $\mathbb{R}^7$ that satifies $\begin{cases} x_1 = x_2 - 2x_4 + x_6 \\ x_3 = -x_4 - x_6 \\ x_5 = 0 \\ x_7 = 0 \end{cases}$, hence

$$\left\{ \boldsymbol{x} \in \mathbb{R}^7 \;\middle|\; \begin{bmatrix} 1 & -1 & 0 & 2 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \boldsymbol{x} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}. \right\}$$

(cartesian equation) □

(L-8) Question 28(c) No, since $\mathcal{C}\left(\mathbf{A}\right) = \mathbb{R}^4$ then $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$ has solution for any vector $\boldsymbol{b}$ in $\mathbb{R}^4$. □

(L-8) Question 28(d)

$$\boldsymbol{b} = \begin{pmatrix} 1 \\ 2 \\ -3 \\ 0 \end{pmatrix}$$

□

(L-8) Question 28(e)

$$\left\{ \boldsymbol{x} \in \mathbb{R}^7 \;\middle|\; \exists\, a, b, c \in \mathbb{R} \text{ such that } \boldsymbol{x} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} + a \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + b \begin{pmatrix} -2 \\ 0 \\ -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + c \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right\}$$

□

(L-8) Question 29(a) El espacio nulo $\mathcal{N}\left(\mathbf{A}\right)$ contiene infinitos vectores, es decir, su dimensión es mayor o igual a uno. Lo sabemos ya que el sistema puede tener infinitas soluciones (una particular mas cualquiera de las del espacio nulo).

□

(L-8) Question 29(b) El espacio $\mathcal{C}\left(\mathbf{A}\right)$ no puede ser todo $\mathbb{R}^m$, pues en ese caso el sistema *siempre* tendría solución, contrariamente a lo que dice el enunciado.

□

(L-8) Question 29(c) Puesto que el sistema puede no tener solución, no todas las filas son pivote (es posible encontrar ecuaciones (0=1)), es decir, que el rango r es menor que el número de filas m.

$$r < m$$

Por otra parte, cuando hay solución, hay infinitas; es decir, el espacio nulo contiene infinitos vectores, por tanto hay columnas libres, es decir, no todas las columnas son pivote.

$$r < n.$$

□

(L-8) Question 29(d) No es posible. Si $\boldsymbol{x}_p$ es solución para el lado derecho $\boldsymbol{b}$, también lo es $\boldsymbol{x}_p + \boldsymbol{x}_n$ para todo vector $\boldsymbol{x}_n$ del espacio nulo $\mathcal{N}\left(\mathbf{A}\right)$.

□

(L-8) Question 30(a)

$$\left[\begin{array}{cccc} 2 & 3 & 1 & -1 \\ 6 & 9 & 3 & -2 \\ \hline 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{\substack{\boldsymbol{\tau} \\ [(2)\mathbf{2}] \\ [(-3)\mathbf{1}+\mathbf{2}] \\ [(2)\mathbf{3}] \\ [(-1)\mathbf{1}+\mathbf{3}] \\ [(2)\mathbf{4}] \\ [(1)\mathbf{1}+\mathbf{4}]}} \left[\begin{array}{cccc} 2 & 0 & 0 & 0 \\ 6 & 0 & 0 & 2 \\ \hline 1 & -3 & -1 & 1 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{array}\right]$$

El espacio nulo es el conjunto de vectores que son combinación lineal de las soluciones especiales; es decir

$$\left\{ \boldsymbol{v} \in \mathbb{R}^4 \;\middle|\; \exists \boldsymbol{p} \in \mathbb{R}^2,\ \boldsymbol{v} = \begin{bmatrix} -3 & -1 \\ 2 & 0 \\ 0 & 2 \\ 0 & 0 \end{bmatrix} \boldsymbol{p} \right\}$$

□

(L-8) Question 30(b) Una solución particular inmediata es el vector

$$\boldsymbol{x}_0 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ -1 \end{pmatrix}$$

(la última columna multiplicada por -1). Así pues, la solución general es cualquier vector $\boldsymbol{x}$ que se pueda expresar como

$$\boldsymbol{x} = \boldsymbol{x}_0 + \boldsymbol{z},$$

donde $\boldsymbol{z}$ es un vector del espacio nulo descrito en el apartado anterior. De forma más explícita

$$\text{Conjunto de vectores: } \left\{ \boldsymbol{x} \in \mathbb{R}^4 \;\middle|\; \exists \boldsymbol{p} \in \mathbb{R}^2 \text{ tal que } \boldsymbol{x} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ -1 \end{pmatrix} + \begin{bmatrix} -1 & -3 \\ 0 & 2 \\ 2 & 0 \\ 0 & 0 \end{bmatrix} \boldsymbol{p} \right\}.$$

□

(L-8) Question 30(c) Cuando una matriz $\underset{m\times n}{\mathbf{A}}$ tiene rango igual a m, la dimensión del espacio columna es m, es decir, $\mathcal{C}\left(\mathbf{A}\right)=\mathbb{R}^m$; y puesto que $\boldsymbol{b}\in\mathbb{R}^m$ el sistema siempre tiene solución, sea cual sea el vector $\boldsymbol{b}\in\mathbb{R}^m$.

El número de soluciones especiales (la dimensión del espacio nulo $\mathcal{N}\left(\mathbf{A}\right)$) es igual al número de columnas libres, es decir, igual a $n-m$ (nótese que sabemos que $n>m$ ya que el rango de matriz es m, si n fuera menor, el rango no podría ser m).

□

(L-8) Question 31. By elimination

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & -1\\ 2 & 2 & -1 & -1\\ 0 & 1 & c & -2 \end{array}\right] \xrightarrow{\substack{\tau\\ [(-1)\mathbf{1}+\mathbf{2}]\\ [(-2)\mathbf{1}+\mathbf{3}]\\ [(1)\mathbf{1}+\mathbf{4}]}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0\\ 2 & 0 & -5 & 1\\ 0 & 1 & c & -2 \end{array}\right] \xrightarrow{\substack{\tau\\ [(-c)\mathbf{2}+\mathbf{3}]\\ [(2)\mathbf{2}+\mathbf{4}]}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0\\ 2 & 0 & -5 & 1\\ 0 & 1 & 0 & 0 \end{array}\right] \xrightarrow{\substack{\tau\\ [(5)\mathbf{4}]\\ [(1)\mathbf{3}+\mathbf{4}]}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0\\ 2 & 0 & -5 & 0\\ 0 & 1 & 0 & 0 \end{array}\right]$$

The system has always only one solution, for any value c.

```
c = sympy.symbols('c')
A = Matrix([[1,1,2],[2,2,-1],[0,1,c]])
b = Vector([-1,-1,-2])
Elim(A.concatena(Matrix([b]),1),1)
```

□

(L-8) Question 32(a)

$$\left[\begin{array}{ccc|c} 1 & -1 & 2 & -1\\ 2 & -3 & m & -3\\ -1 & 2 & 3 & -2m\\ \hline 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ \hline 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{\substack{\tau\\ [(1)\mathbf{1}+\mathbf{2}]\\ [(-2)\mathbf{1}+\mathbf{3}]}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & -1\\ 2 & -1 & m-4 & -3\\ -1 & 1 & 5 & -2m\\ \hline 1 & 1 & -2 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ \hline 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{\substack{\tau\\ [(-1)\mathbf{2}]\\ [(-2)\mathbf{2}+\mathbf{1}]\\ [(-m+4)\mathbf{2}+\mathbf{3}]}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & -1\\ 0 & 1 & 0 & -3\\ 1 & -1 & m+1 & -2m\\ \hline 3 & -1 & m-6 & 0\\ 2 & -1 & m-4 & 0\\ 0 & 0 & 1 & 0\\ \hline 0 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{\substack{\tau\\ [(1)\mathbf{1}+\mathbf{4}]\\ [(3)\mathbf{2}+\mathbf{4}]}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 1 & -1 & m+1 & -2(m+1)\\ \hline 3 & -1 & m-6 & 0\\ 2 & -1 & m-4 & -1\\ 0 & 0 & 1 & 0\\ \hline 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{\substack{\tau\\ [(2)\mathbf{3}+\mathbf{4}]}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 1 & -1 & m+1 & 0\\ \hline 3 & -1 & m-6 & 2(m-6)\\ 2 & -1 & m-4 & 2(m-4)-1\\ 0 & 0 & 1 & 2\\ \hline 0 & 0 & 0 & 1 \end{array}\right]$$

□

(L-8) Question 32(b)

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 1 & -1 & 0 & 0\\ \hline 3 & -1 & -7 & 0\\ 2 & -1 & -5 & -1\\ 0 & 0 & 1 & 0\\ \hline 0 & 0 & 0 & 1 \end{array}\right]$$

Una solución particular es $\boldsymbol{x}_p=\begin{pmatrix}0\\-1\\0\end{pmatrix}$, y las soluciones al sistema homogéneo son los múltiplos del vector $\boldsymbol{x}_n=\begin{pmatrix}-7\\-5\\1\end{pmatrix}$. Por tanto, la solución completa al sistema son todos los vectores que se pueden escribir como $\boldsymbol{x}=\boldsymbol{x}_p+a\boldsymbol{x}_n$ para cualquier número real a.

□

(L-8) Question 32(c) El conjunto de puntos que son solución al sistema del apartado anterior es una recta en $\mathbb{R}^3$.

No es posible que el conjunto de soluciones sea un plano en ningún caso; para que fuera posible sería necesario que la matriz de coeficientes del sistema fuera de rango 1. Pero en este caso el rango es 2 para $m = -1$ o rango 3 cuando $m \neq -1$. En este último caso (rango 3), el conjunto de soluciones es un punto en $\mathbb{R}^3$.

□

(L-8) Question 32(d) En este caso tenemos

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & -1 & 2 & 0 \\ \hline 3 & -1 & -5 & -10 \\ 2 & -1 & -3 & -7 \\ 0 & 0 & 1 & 2 \\ \hline 0 & 0 & 0 & 1 \end{array}\right]$$

Por tanto la solución en este caso es

$$\boldsymbol{x} = \begin{pmatrix} -10 \\ -7 \\ 2 \end{pmatrix}.$$

□

(L-8) Question 33(a)

$$\left[\begin{array}{ccc} 1 & 1 & 1 \\ 1 & 0 & 2 \\ \hline 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right] \xrightarrow[{[(-1)\mathbf{1}+\mathbf{3}]}]{\substack{\tau \\ [(-1)\mathbf{1}+\mathbf{2}]}} \left[\begin{array}{ccc} 1 & 0 & 0 \\ 1 & -1 & 1 \\ \hline 1 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right] \xrightarrow{\substack{\tau \\ [(1)\mathbf{2}+\mathbf{3}]}} \left[\begin{array}{ccc} 1 & 0 & 0 \\ 1 & -1 & 0 \\ \hline 1 & -1 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{array}\right]$$

False. Since the solutions are multiples of

$$\boldsymbol{x}_0 = \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix};$$

the set of solutions is the whole line $c\boldsymbol{x}_0$ for all c.

□

(L-8) Question 33(b) True

□

(L-8) Question 33(c) False

□

(L-8) Question 33(d) True. It is a line through the origin.

□

(L-8) Question 33(e) True.

□

(L-8) Question 33(f) False.

□

(L-8) Question 34(f)

$$\left[\begin{array}{cc|c} 1 & 0 & -b_1 \\ 4 & 1 & -b_2 \\ 2 & -1 & -b_3 \\ \hline 1 & 0 & 0 \\ 0 & 1 & 0 \\ \hline 0 & 0 & 1 \end{array}\right] \xrightarrow[{[(b_2)\mathbf{2}+\mathbf{3}]}]{\substack{\tau \\ [(-4)\mathbf{2}+\mathbf{1}]}} \left[\begin{array}{cc|c} 1 & 0 & -b_1 \\ 0 & 1 & 0 \\ 6 & -1 & -b_2 - b_3 \\ \hline 1 & 0 & 0 \\ -4 & 1 & b_2 \\ \hline 0 & 0 & 1 \end{array}\right] \xrightarrow{\substack{\tau \\ [(b_1)\mathbf{1}+\mathbf{3}]}} \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 6 & -1 & 6b_1 - b_2 - b_3 \\ \hline 1 & 0 & b_1 \\ -4 & 1 & b_2 - 4b_1 \\ \hline 0 & 0 & 1 \end{array}\right]$$

Any vector in $\mathbb{R}^3$ such that $6b_1 - b_2 - b_3 = 0$.

□

(L-8) Question 35(a)

```
A = Matrix( [ [0,3,3,0], [4,3,0,0], [0,0,2,2] ] )
b = Vector( [39, 44, 22] )
SEL(A,b,1)
```

$$\left[\begin{array}{cccc|c} 0 & 3 & 3 & 0 & -39 \\ 4 & 3 & 0 & 0 & -44 \\ 0 & 0 & 2 & 2 & -22 \\ \hline 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{cccc|c} 0 & 12 & 0 & 0 & 0 \\ 4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 8 & 0 & 0 \\ \hline 1 & -3 & 3 & -3 & 19/2 \\ 0 & 4 & -4 & 4 & 2 \\ 0 & 0 & 4 & -4 & 11 \\ 0 & 0 & 0 & 4 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

Puesto que el rango es tres (igual al número de filas) siempre hay solución al sistema; pero como hay columnas libres, el sistema tiene infinitas soluciones. (Nota: no es necesario llegar a la forma escalonada reducida, con llegar a la forma pre-escalonada $\mathbf{K}$, es suficiente para ver que hay tres pivotes y una columna libre. Con los pasos anteriores hemos obtenido una solución particular, pero tampoco esto es necesario para responder a la pregunta; bastaba trabajar con la matriz de coeficientes $\mathbf{A}$ y mirar su rango.). □

(L-8) Question 35(b) En este caso el sistema se reduce a

$$\begin{bmatrix} 3 & 3 & 0 \\ 3 & 4 & 0 \\ 0 & 2 & 2 \end{bmatrix} \begin{pmatrix} a \\ i \\ d \end{pmatrix} = \begin{pmatrix} 39 \\ 44 \\ 22 \end{pmatrix}$$

donde en este caso i hace referencia al precio tanto de la entrada infantil como a la de tercera edad (que son iguales).

$$\left[\begin{array}{ccc|c} 3 & 3 & 0 & -39 \\ 3 & 4 & 0 & -44 \\ 0 & 2 & 2 & -22 \\ \hline 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{\substack{\tau \\ {[(-1)\mathbf{1}+\mathbf{2}]} \\ {[(13)\mathbf{1}+\mathbf{4}]}}} \left[\begin{array}{ccc|c} 3 & 0 & 0 & 0 \\ 3 & 1 & 0 & -5 \\ 0 & 2 & 2 & -22 \\ \hline 1 & -1 & 0 & 13 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{\substack{\tau \\ {[(5)\mathbf{2}+\mathbf{4}]} \\ {[(6)\mathbf{3}+\mathbf{4}]}}} \left[\begin{array}{ccc|c} 3 & 0 & 0 & 0 \\ 3 & 1 & 0 & 0 \\ 0 & 2 & 2 & 0 \\ \hline 1 & -1 & 0 & 8 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 6 \\ \hline 0 & 0 & 0 & 1 \end{array}\right]$$

Por tanto los precios son 8 para adultos, 5 para infantiles (y tercera edad) y 6 la tarifa reducida.

□

(L-8) Question 36(a)

$$\left[\begin{array}{cccc|c} 1 & 2 & -1 & 1 & 1 \\ -1 & -2 & 3 & 5 & 5 \\ -1 & -2 & -1 & -7 & -7 \\ \hline 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{\substack{\tau \\ {[(-2)\mathbf{1}+\mathbf{2}]} \\ {[(1)\mathbf{1}+\mathbf{3}]} \\ {[(-1)\mathbf{1}+\mathbf{4}]} \\ {[(-1)\mathbf{1}+\mathbf{5}]}}} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 2 & 6 & 6 \\ -1 & 0 & -2 & -6 & -6 \\ \hline 1 & -2 & 1 & -1 & -1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{\substack{\tau \\ {[(-3)\mathbf{3}+\mathbf{4}]} \\ {[(-3)\mathbf{3}+\mathbf{5}]}}} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 2 & 0 & 0 \\ -1 & 0 & -2 & 0 & 0 \\ \hline 1 & -2 & 1 & -4 & -4 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -3 & -3 \\ 0 & 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

Tanto la matriz de coeficientes, cómo la matriz ampliada tienen rango 2.

□

(L-8) Question 36(b) Una solución particular es

$$\boldsymbol{x}_p = \begin{pmatrix} -4 \\ 0 \\ -3 \\ 0 \end{pmatrix},$$

Solución al sistema homogéneo es el conjunto de combinaciones lineales de los vectores

$$\boldsymbol{x}_a = \begin{pmatrix}-2\\1\\0\\0\end{pmatrix} \quad \text{y} \quad \boldsymbol{x}_b = \begin{pmatrix}-4\\0\\-3\\1\end{pmatrix},$$

Así pues, la solución al sistema propuesto es

$$\left\{ \boldsymbol{x}\in\mathbb{R}^4 \;\middle|\; \exists\, a,b\in\mathbb{R} \text{ such that } \boldsymbol{x} = \begin{pmatrix}-4\\0\\-3\\0\end{pmatrix} + a\begin{pmatrix}-2\\1\\0\\0\end{pmatrix} + b\begin{pmatrix}-4\\0\\-3\\1\end{pmatrix} \right\}$$

□

(L-8) Question 36(c) Es un **plano** paralelo al generado por las combinaciones lineales de $\boldsymbol{x}_a$ y $\boldsymbol{x}_b$ (que es la solución del sistema homogéneo) pero que pasa por el punto $\boldsymbol{x}_p = (-4, 0, -3, 0,)$ (que es uno de los infinitos vectores que resuelven el sistema completo).

□

(L-8) Question 37(a) We need a rank 3 matrix; by Gaussian elimination we get:

$$\left[\begin{array}{cccc}1&0&1&1\\1&1&2&3\\a&1&1&2\\\hline 1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{array}\right] \xrightarrow[{[(-1)\mathbf{1}+\mathbf{4}]}]{\substack{\tau\\{[(-1)\mathbf{1}+\mathbf{3}]}}} \left[\begin{array}{cccc}1&0&0&0\\1&1&1&2\\a&1&1-a&2-a\\\hline 1&0&-1&-1\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{array}\right] \xrightarrow[{[(-1)\mathbf{2}+\mathbf{4}]}]{\substack{\tau\\{[(-1)\mathbf{2}+\mathbf{3}]}}} \left[\begin{array}{cccc}1&0&0&0\\1&1&0&0\\a&1&-a&-a\\\hline 1&0&-1&-1\\0&1&-1&-2\\0&0&1&0\\0&0&0&1\end{array}\right] \xrightarrow[{[(-1)\mathbf{3}+\mathbf{4}]}]{\tau} \left[\begin{array}{cccc}1&0&0&0\\1&1&0&0\\a&1&-a&0\\\hline 1&0&-1&0\\0&1&-1&-1\\0&0&1&-1\\0&0&0&1\end{array}\right]$$

therefore, if $a \neq 0$ the rank of $\mathbf{A}$ is 3, and the dimension of $\mathcal{N}(\mathbf{A})$ is one.

□

(L-8) Question 37(b) When $a = 0$; in that case $\dim\mathcal{N}(\mathbf{A}) = 2$.

□

(L-8) Question 38.

```
A = Matrix([[1,3,2,4,-3], [2,6,0,-1,-2], [0,0,6,2,-1], [1,3,-1,4,2]])
b = Vector([-7,0,12,-6])
SEL(A,b,1)
```

$$\text{Conjunto de vectores: } \left\{ \boldsymbol{x}\in\mathbb{R}^5 \;\middle|\; \exists \boldsymbol{p}\in\mathbb{R}^1 \text{ tal que } \boldsymbol{x} = \begin{pmatrix}1\\0\\3\\-2\\2\end{pmatrix} + \begin{bmatrix}-3\\1\\0\\0\\0\end{bmatrix}\boldsymbol{p} \right\}$$

□

(L-8) Question 39(a)

$$\left[\begin{array}{c|c}\mathsf{A} & -\boldsymbol{b}\\ \hline \mathsf{I} & \mathbf{0}\\ \hline \mathbf{0} & 1\end{array}\right]=\left[\begin{array}{cccc|c}1&3&2&2&-2\\2&7&6&8&-7\\3&9&6&7&-7\\ \hline 1&0&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\ \hline 0&0&0&0&1\end{array}\right]\xrightarrow[{[(2)\mathbf{1}+\mathbf{5}]}]{\substack{\tau\\ {[(-3)\mathbf{1}+\mathbf{2}]}\\ {[(-2)\mathbf{1}+\mathbf{3}]}\\ {[(-2)\mathbf{1}+\mathbf{4}]}}}\left[\begin{array}{cccc|c}1&0&0&0&0\\2&1&2&4&-3\\3&0&0&1&-1\\ \hline 1&-3&-2&-2&2\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\ \hline 0&0&0&0&1\end{array}\right]$$

$$\xrightarrow[{[(3)\mathbf{2}+\mathbf{5}]}]{\substack{\tau\\ {[(-2)\mathbf{2}+\mathbf{3}]}\\ {[(-4)\mathbf{2}+\mathbf{4}]}}}\left[\begin{array}{cccc|c}1&0&0&0&0\\2&1&0&0&0\\3&0&0&1&-1\\ \hline 1&-3&4&10&-7\\0&1&-2&-4&3\\0&0&1&0&0\\0&0&0&1&0\\ \hline 0&0&0&0&1\end{array}\right]\xrightarrow{\substack{\tau\\ {[(1)\mathbf{4}+\mathbf{5}]}}}\left[\begin{array}{cccc|c}1&0&0&0&0\\2&1&0&0&0\\3&0&0&1&0\\ \hline 1&-3&4&10&3\\0&1&-2&-4&-1\\0&0&1&0&0\\0&0&0&1&1\\ \hline 0&0&0&0&1\end{array}\right]=\left[\begin{array}{c|c}\mathsf{L} & \mathbf{0}\\ \hline \mathsf{E} & \boldsymbol{x}_p\\ \hline \mathbf{0} & 1\end{array}\right]$$

La solución es el subconjunto de vectores de $\mathbb{R}^4$

$$\left\{ \boldsymbol{x}\in\mathbb{R}^4 \;\middle|\; \exists\, a\in\mathbb{R} \text{ such that } \boldsymbol{x}=\begin{pmatrix}3\\-1\\0\\1\end{pmatrix}+a\begin{pmatrix}4\\-2\\1\\0\end{pmatrix}\right\}$$

Dicho conjunto de soluciones es una recta, pero puesto que no pasa por el origen de coordenadas (no contiene a $\mathbf{0}$), no es espacio vectorial. Es decir, es una recta en la dirección del vector del espacio nulo $\boldsymbol{x}_a$, que pasa por el punto $\boldsymbol{x}_p$, pero no por el origen.

□

(L-8) Question 39(b) Puesto que hay tres columnas pivote (tres columnas linealmente independientes), el espacio columna es todo el espacio $\mathbb{R}^3$.

Si en lugar de un 7 hubiera un 6, en el segundo paso de eliminación generaríamos una fila de ceros, y por tanto habría sólo dos pivotes y $\mathcal{C}\left(\mathsf{A}\right)$ sería tan sólo un plano dentro de $\mathbb{R}^3$ que pasa por el origen.

$$\left[\begin{array}{c|c}\mathsf{M} & -\boldsymbol{b}\\ \hline \mathsf{I} & \mathbf{0}\\ \hline \mathbf{0} & 1\end{array}\right]=\left[\begin{array}{cccc|c}1&3&2&2&-2\\2&7&6&8&-7\\3&9&6&6&-7\\ \hline 1&0&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\ \hline 0&0&0&0&1\end{array}\right]\xrightarrow[{[(2)\mathbf{1}+\mathbf{5}]}]{\substack{\tau\\ {[(-3)\mathbf{1}+\mathbf{2}]}\\ {[(-2)\mathbf{1}+\mathbf{3}]}\\ {[(-2)\mathbf{1}+\mathbf{4}]}}}\left[\begin{array}{cccc|c}1&0&0&0&0\\2&1&2&4&-3\\3&0&0&0&-1\\ \hline 1&-3&-2&-2&2\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\ \hline 0&0&0&0&1\end{array}\right]\xrightarrow[{[(3)\mathbf{2}+\mathbf{5}]}]{\substack{\tau\\ {[(-2)\mathbf{2}+\mathbf{3}]}\\ {[(-4)\mathbf{2}+\mathbf{4}]}}}\left[\begin{array}{cccc|c}1&0&0&0&0\\2&1&0&0&0\\3&0&0&0&-1\\ \hline 1&-3&4&10&-7\\0&1&-2&-4&3\\0&0&1&0&0\\0&0&0&1&0\\ \hline 0&0&0&0&1\end{array}\right]$$

□

(L-8) Question 39(c) La primera parte es sencilla, basta elegir como lado derecho $\boldsymbol{b}$ cualquiera de las columnas de M. La segunda parte también es fácil empleando lo que sabemos. Si mantenemos el vector derecho $\boldsymbol{b}$ del enunciado, el sistema no tiene solución.

□

(L-9) Question 1.

$$\left[\begin{array}{cccc} 1 & 1 & 0 & 0\\ 1 & 0 & 0 & 1\\ 0 & 1 & 1 & 0\\ 0 & 0 & 1 & 1\\ \hline 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{\underset{[(-1)\mathbf{1}+\mathbf{2}]}{\boldsymbol{\tau}}} \left[\begin{array}{cccc} 1 & 0 & 0 & 0\\ 1 & -1 & 0 & 1\\ 0 & 1 & 1 & 0\\ 0 & 0 & 1 & 1\\ \hline 1 & -1 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{\underset{[(1)\mathbf{2}+\mathbf{4}]}{\boldsymbol{\tau}}} \left[\begin{array}{cccc} 1 & 0 & 0 & 0\\ 1 & -1 & 0 & 0\\ 0 & 1 & 1 & 1\\ 0 & 0 & 1 & 1\\ \hline 1 & -1 & 0 & -1\\ 0 & 1 & 0 & 1\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{\underset{[(-1)\mathbf{3}+\mathbf{4}]}{\boldsymbol{\tau}}} \left[\begin{array}{cccc} 1 & 0 & 0 & 0\\ 1 & -1 & 0 & 0\\ 0 & 1 & 1 & 0\\ 0 & 0 & 1 & 0\\ \hline 1 & -1 & 0 & -1\\ 0 & 1 & 0 & 1\\ 0 & 0 & 1 & -1\\ 0 & 0 & 0 & 1 \end{array}\right]$$

Since there are only three pivots, the four columns are linearly dependent (in fact if we add the first and third columns and then we substract the second one we get the fourth).

Extending the matrix with the new column $\begin{pmatrix}0, & 0, & 0, & 1,\end{pmatrix}$, we have

$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 0\\ 1 & 0 & 0 & 1 & 0\\ 0 & 1 & 1 & 0 & 0\\ 0 & 0 & 1 & 1 & -1\\ \hline 1 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1 & 0\\ \hline 0 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{\substack{\boldsymbol{\tau}\\ [(-1)\mathbf{1}+\mathbf{2}]\\ [(1)\mathbf{2}+\mathbf{4}]\\ [(-1)\mathbf{3}+\mathbf{4}]}} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0\\ 1 & -1 & 0 & 0 & 0\\ 0 & 1 & 1 & 0 & 0\\ 0 & 0 & 1 & 0 & -1\\ \hline 1 & -1 & 0 & -1 & 0\\ 0 & 1 & 0 & 1 & 0\\ 0 & 0 & 1 & -1 & 0\\ 0 & 0 & 0 & 1 & 0\\ \hline 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

It is not possible transforming the right hand side vector in a column of zeros by column elimination. Therefore, the system has no solution. There isn't any linear combination of the columns that reaches the right hand side vector in $\mathbb{R}^4$. Therefore, those four columns don't span $\mathbb{R}^4$.

□

(L-9) Question 2(a) *might not* span (if there is not a subset of 4 independent vectors).

□

(L-9) Question 2(b) Those vectors *are not* linearly independent (there are more than four!).

□

(L-9) Question 2(c) The system *might not have* a solution (if $\boldsymbol{b}$ isn't a linear combination of the columns of $\mathbf{A}$).

□

(L-9) Question 2(d) $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$ *does not have* a sole solution. There are more unknows (6) than equations(4), therefore there is no solution to the system at all, or there are infinite solutions (but never only one!).

□

(L-9) Question 3(a) Matrix $\mathbf{B}$ should have order 3×2. We also know there is only one solution to the system (the complete solution consists of only one vector), and the null space consists of only the zero vector. Therefore the rank is two (both columns are independent). One example of such a matrix is

$$\mathbf{B} = \begin{bmatrix} 1 & 1\\ 2 & 0\\ 4 & 0 \end{bmatrix}$$

but also any other matrix such as

$$\mathbf{B} = \begin{bmatrix} 1 & a\\ 2 & b\\ 4 & c \end{bmatrix}$$

if both columns are independent.

□

(L-9) Question 3(b) There is not such a matrix:

On the one hand the order of $\mathbf{C}$ should be 2×3. (two equations and three unknowns), and then, at least one column is free. Therefore there will be infinte solutions.

But on the other hand the exercise claims that the complete solution is only one vector... that is impossible.

$\square$

(L-9) Question 4. A system of vectors is independent when $x_1\boldsymbol{v}_1 + x_2\boldsymbol{v}_2 + \cdots + x_k\boldsymbol{v}_k = \mathbf{0}$ if and only if $x_1 = x_2 = \cdots = x_k = 0$ (no free columns).

In the first case:

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \mathbf{0}$$

has the only solution $x_1 = x_2 = x_3 = 0$. Therefore, these three vectors are independent.

But in the second case

$$\begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 1 & 4 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \mathbf{0}$$

is a system with infinite solutions; for example $\boldsymbol{x} = \begin{pmatrix} 1 \\ 1 \\ -4 \\ 1 \end{pmatrix}$. So $\boldsymbol{v}_1 + \boldsymbol{v}_2 - 4\boldsymbol{v}_3 + \boldsymbol{x}_4 = \mathbf{0}$, and then, the four vectors are dependent.

$\square$

(L-9) Question 5. If $\mathbf{A}^{\intercal} = 2\mathbf{A}$, then also $\mathbf{A} = 2\mathbf{A}^{\intercal} = 2(2\mathbf{A}) = 4\mathbf{A}$ so $\mathbf{A} = \mathbf{0}$; and, of course, the rows of $\mathbf{A}$ are then linearly dependent.

$\square$

(L-9) Question 6(a) **No, they don't span.** only two vectors can't span the three dimensional space $\mathbb{R}^3$.

$\square$

(L-9) Question 6(b) **Yes.** Since we can find three pivots, we known that these three vectors are linearly independent. We can always find a solution to $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$ for any vector $\boldsymbol{b}$ in $\mathbb{R}^3$, $\quad (\mathcal{C}\left(\mathbf{A}\right) = \mathbb{R}^3)$.

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 3 \\ 0 & -2 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \mathbf{R}$$

$\square$

(L-9) Question 6(c) **No, they don't span.** We can only find two pivots. We can't find a solution to $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$ for some vectors $\boldsymbol{b}$ in $\mathbb{R}^3$, $\quad (\mathcal{C}\left(\mathbf{A}\right) \neq \mathbb{R}^3)$.

$\square$

(L-9) Question 6(d) **Yes.** Since we can find three pivots, $\quad (\mathcal{C}\left(\mathbf{A}\right) = \mathbb{R}^3)$.

$\square$

(L-9) Question 7(a) **Dependent.** Solving the linear system

$$a\begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix} + b\begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 4 \\ 7 \\ 3 \end{pmatrix}$$

$$\left[\begin{array}{cc|c} -1 & 2 & -4 \\ 2 & 1 & -7 \\ 3 & -1 & -3 \\ \hline 1 & 0 & 0 \\ 0 & 1 & 0 \\ \hline 0 & 0 & 1 \end{array}\right] \xrightarrow[{[(2)\mathbf{1}+\mathbf{2}]}]{\boldsymbol{\tau}} \left[\begin{array}{cc|c} -1 & 0 & -4 \\ 2 & 5 & -7 \\ 3 & 5 & -3 \\ \hline 1 & 2 & 0 \\ 0 & 1 & 0 \\ \hline 0 & 0 & 1 \end{array}\right] \xrightarrow[{[(-4)\mathbf{1}+\mathbf{3}]}]{\boldsymbol{\tau}} \left[\begin{array}{cc|c} -1 & 0 & 0 \\ 2 & 5 & -15 \\ 3 & 5 & -15 \\ \hline 1 & 2 & -4 \\ 0 & 1 & 0 \\ \hline 0 & 0 & 1 \end{array}\right] \xrightarrow[{[(3)\mathbf{2}+\mathbf{3}]}]{\boldsymbol{\tau}} \left[\begin{array}{cc|c} -1 & 0 & 0 \\ 2 & 5 & 0 \\ 3 & 5 & 0 \\ \hline 1 & 2 & 2 \\ 0 & 1 & 3 \\ \hline 0 & 0 & 1 \end{array}\right]$$

we find that $2(-1, 2, 3,) + 3(2, 1, -1,) = (4, 7, 3,)$. Three vectors and only two pivots!

$\square$

(L-9) Question 7(b) **Independent.** Two vectors, two pivots.

$$\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 2 & -1 \\ 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} = \mathbf{R}$$

$\square$

(L-9) Question 7(c) **Dependent.** Three vectors, two pivots

$$\left[\begin{array}{cc|c} 1 & 2 & -8 \\ 2 & 3 & 2 \\ \hline 1 & 0 & 0 \\ 0 & 1 & 0 \\ \hline 0 & 0 & 1 \end{array}\right] \xrightarrow{\substack{\tau \\ [(-2)\mathbf{1}+\mathbf{2}]}} \left[\begin{array}{cc|c} 1 & 0 & -8 \\ 2 & -1 & 2 \\ \hline 1 & -2 & 0 \\ 0 & 1 & 0 \\ \hline 0 & 0 & 1 \end{array}\right] \xrightarrow{\substack{\tau \\ [(8)\mathbf{1}+\mathbf{3}]}} \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 2 & -1 & 18 \\ \hline 1 & -2 & 8 \\ 0 & 1 & 0 \\ \hline 0 & 0 & 1 \end{array}\right] \xrightarrow{\substack{\tau \\ [(18)\mathbf{2}+\mathbf{3}]}} \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 2 & -1 & 0 \\ \hline 1 & -2 & -28 \\ 0 & 1 & 18 \\ \hline 0 & 0 & 1 \end{array}\right]$$

then $18(2, 3,) - 28(1, 2,) = (8, -2,)$.

$\square$

(L-9) Question 7(d) **Dependent.** The fourth vector (polynomial) is the sum of the other three. Using the coeficients of the polynomials as vectors in $\mathbb{R}^4$ we can solve the system:

$$\left[\begin{array}{ccc|c} 0 & 1 & 1 & -1 \\ 1 & -1 & 0 & 0 \\ 2 & 0 & 0 & -1 \\ 1 & 0 & 1 & -1 \\ \hline 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{\substack{\tau \\ [(-1)\mathbf{2}+\mathbf{3}] \\ [(1)\mathbf{2}+\mathbf{4}] \\ [(1)\mathbf{1}+\mathbf{2}] \\ [(-1)\mathbf{1}+\mathbf{3}] \\ [(1)\mathbf{1}+\mathbf{4}]}} \left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 2 & 2 & -2 & 1 \\ 1 & 1 & 0 & 0 \\ \hline 1 & 1 & -1 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{\substack{\tau \\ [(2)\mathbf{4}] \\ [(1)\mathbf{3}+\mathbf{4}]}} \left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 2 & 2 & -2 & 0 \\ 1 & 1 & 0 & 0 \\ \hline 1 & 1 & -1 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 1 \\ \hline 0 & 0 & 0 & 2 \end{array}\right] \xrightarrow{\substack{\tau \\ [(\frac{1}{2})\mathbf{4}]}} \left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 2 & 2 & -2 & 0 \\ 1 & 1 & 0 & 0 \\ \hline 1 & 1 & -1 & 1/2 \\ 0 & 1 & -1 & 1/2 \\ 0 & 0 & 1 & 1/2 \\ \hline 0 & 0 & 0 & 1 \end{array}\right]$$

finding the same result: the last polynomial is half the sum of the other three.

$\square$

(L-9) Question 8. That means the null space has dimension 0 (there aren't any free columns). All the n columns are pivot columns (rank n). All columns are linearly independent.

$\square$

(L-9) Question 9. When the columns are linear independent, it is no possible to get a zero column by Gauss-Jordan column elimination. Hence the colums are linear independent if and only all columns are pivot columns in any (pre)echelon form of the matrix. But each pivot apears in a different row, and since there are only 4 rows, there are, at most, 4 pivots. Thus, at least two columns are free columns.

$\square$

(L-9) Question 10(a) Since the nullspace is spanned by the given three vectors, we may simply take $\mathbf{B}$ to consist of the three vectors as columns, i.e.,

$$\mathbf{B} = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & -1 \\ -1 & 1 & 3 \\ 3 & 4 & 1 \end{bmatrix}$$

$\mathbf{B}$ need not be square.

$\square$

(L-9) Question 10(b) For example, we may simply add a zero column to $\mathbf{B}$:

$$\mathbf{B} = \begin{bmatrix} 0 & 1 & 0 & -1 \\ 0 & 2 & 1 & -1 \\ 0 & -1 & 1 & 3 \\ 0 & 3 & 4 & 1 \end{bmatrix}$$

Or, we could interchange two columns. Or we could multiply one of the columns by -1. For example:

$$\mathbf{B} = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 1 \\ -1 & 1 & -3 \\ 3 & 4 & -1 \end{bmatrix}$$

Or we could replace one of the columns by a linear combination of that column with the other two columns (any invertible column operation). Or we could replace $\mathbf{B}$ by $-\mathbf{B}$ or $2\mathbf{B}$. There are many possible solutions. In any case, the solution shouldn't require any significant calculation!

□

(L-9) Question 10(c) Since any solution $\boldsymbol{x}$ to the equation $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$ is of the form $\boldsymbol{x}_p + \boldsymbol{n}$ for some vector $\boldsymbol{n}$ in the nullspace, the vector $\boldsymbol{x} - \boldsymbol{x}_p$ must lie in the nullspace $\mathcal{N}(\mathbf{A})$. Thus, we want to look at:

$$\boldsymbol{x}_Z - \boldsymbol{x}_p = \begin{pmatrix} 0 \\ -1 \\ 0 \\ -4 \end{pmatrix}, \quad \boldsymbol{x} - \boldsymbol{x}_p \begin{pmatrix} 0 \\ -1 \\ 0 \\ -3 \end{pmatrix}.$$

To determine whether a vector $\boldsymbol{y}$ lies in the nullspace $\mathcal{N}(\mathbf{A})$, we can just check whether it is in the column space of $\mathbf{B}$, i.e. check whether $\mathbf{B}\boldsymbol{z} = \boldsymbol{y}$ has a solution. As we learned in class, we can check this just by doing elimination:

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 2 & 1 & -1 & -1 \\ -1 & 1 & 3 & 0 \\ 3 & 4 & 1 & -a \\ \hline 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{\substack{\boldsymbol{\tau} \\ [(1)\mathbf{1}+\mathbf{3}] \\ [(-1)\mathbf{2}+\mathbf{3}] \\ [(1)\mathbf{2}+\mathbf{4}]}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ -1 & 1 & 1 & 1 \\ 3 & 4 & 0 & -a-4 \\ \hline 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{\substack{\boldsymbol{\tau} \\ [(-1)\mathbf{3}+\mathbf{4}]}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ -1 & 1 & 1 & 0 \\ 3 & 4 & 0 & -a-4 \\ \hline 1 & 0 & 1 & -1 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 1 & -1 \\ \hline 0 & 0 & 0 & 1 \end{array}\right]$$

We can get a solution if and only if $a = -4$. So Zarkon is correct.

□

(L-9) Question 11(a)

$$\left[\begin{array}{ccccc|c} 1 & 2 & -1 & 0 & 0 & -a \\ 1 & 2 & 0 & 2 & 2 & -b \\ 1 & 2 & -1 & 0 & 0 & -c \\ 2 & 4 & 0 & 4 & 4 & -d \\ \hline 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{\substack{\boldsymbol{\tau} \\ [(-2)\mathbf{1}+\mathbf{2}] \\ [(1)\mathbf{1}+\mathbf{3}] \\ [(a)\mathbf{1}+\mathbf{6}]}} \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 2 & 2 & a-b \\ 1 & 0 & 0 & 0 & 0 & a-c \\ 2 & 0 & 2 & 4 & 4 & 2a-d \\ \hline 1 & -2 & 1 & 0 & 0 & a \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{\substack{\boldsymbol{\tau} \\ [(-2)\mathbf{3}+\mathbf{4}] \\ [(-2)\mathbf{3}+\mathbf{5}] \\ [(b-a)\mathbf{3}+\mathbf{6}]}} \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & a-c \\ 2 & 0 & 2 & 0 & 0 & 2b-d \\ \hline 1 & -2 & 1 & -2 & -2 & b \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -2 & -2 & b-a \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

Therefore

$$\left[\begin{pmatrix} 1 \\ 1 \\ 1 \\ 2 \end{pmatrix}; \begin{pmatrix} -1 \\ 0 \\ -1 \\ 0 \end{pmatrix}; \right] \quad \text{and also} \quad \left[\begin{pmatrix} 1 \\ 1 \\ 1 \\ 2 \end{pmatrix}; \begin{pmatrix} 0 \\ 1 \\ 0 \\ 2 \end{pmatrix}; \right].$$

□

(L-9) Question 11(b)

$$\left[\begin{pmatrix}-2\\1\\0\\0\\0\end{pmatrix};\quad\begin{pmatrix}-2\\0\\-2\\1\\0\end{pmatrix};\quad\begin{pmatrix}-2\\0\\-2\\0\\1\end{pmatrix};\right]$$

□

(L-9) Question 11(c) $c-a=0$ and $d-2b=0$.

□

(L-9) Question 11(d)

$$\left\{\boldsymbol{x}\in\mathbb{R}^5 \;\middle|\; \exists\, a,b,c\in\mathbb{R} \text{ such that } \boldsymbol{x}=\begin{pmatrix}1\\0\\1\\0\\0\end{pmatrix}+a\begin{pmatrix}-2\\1\\0\\0\\0\end{pmatrix}+b\begin{pmatrix}-2\\0\\-2\\1\\0\end{pmatrix}+c\begin{pmatrix}-2\\0\\-2\\0\\1\end{pmatrix}\right\}.$$

□

(L-9) Question 12. Si a una matriz A se le "añade" una nueva columna extra $\boldsymbol{b}$, entonces el espacio columna se vuelve más grande, a no ser que el vector $\boldsymbol{b}$ ya esté en $\mathcal{C}\left(\mathsf{A}\right)$.

Caso en que se hace más grande

$$\mathsf{A}=\begin{bmatrix}1&0\\0&0\end{bmatrix};\qquad \boldsymbol{b}=\begin{pmatrix}1\\1\end{pmatrix};\qquad [\mathsf{A}\ \boldsymbol{b}]=\begin{bmatrix}1&0&1\\0&0&1\end{bmatrix}.$$

En este caso $\mathcal{C}\left([\mathsf{A}\ \boldsymbol{b}]\right)$ es más grande que $\mathcal{C}\left(\mathsf{A}\right)$; y el sistema $\mathsf{A}\boldsymbol{x}=\boldsymbol{b}$ no tiene solución por NO pertenecer $\boldsymbol{b}$ al espacio columna de A (es decir, porque $\boldsymbol{b}\notin\mathcal{C}\left(\mathsf{A}\right)$). Nótese que en este caso ninguna combinación lineal de las columnas de A puede ser igual a $\boldsymbol{b}$. Nótese que el rango de A es 1, pero el de la matriz ampliada $[\mathsf{A}\mid -\boldsymbol{b}]$ es 2, así que el método de Gauss visto en clase fallaría.

Caso en el que es igual de grande

$$\mathsf{A}=\begin{bmatrix}1&0\\0&1\end{bmatrix};\qquad \boldsymbol{b}=\begin{pmatrix}1\\1\end{pmatrix};\qquad [\mathsf{A}\ \boldsymbol{b}]=\begin{bmatrix}1&0&1\\0&1&1\end{bmatrix}.$$

En este caso $\mathcal{C}\left([\mathsf{A}\ \boldsymbol{b}]\right)=\mathcal{C}\left(\mathsf{A}\right)$; y el sistema $\mathsf{A}\boldsymbol{x}=\boldsymbol{b}$ tiene solución por pertenecer $\boldsymbol{b}$ al espacio columna de A (es decir, porque $\boldsymbol{b}\in\mathcal{C}\left(\mathsf{A}\right)$).

□

(L-9) Question 13. $\mathcal{C}\left(\mathsf{A}\right)=\mathbb{R}^9$.

□

(L-9) Question 14. Puesto que todo vector $\boldsymbol{a}$ en $\mathcal{V}$, se puede expresar como $x_1\boldsymbol{v}_1+\cdots+x_n\boldsymbol{v}_n$; es decir como

$$\boldsymbol{a}=\begin{bmatrix}\boldsymbol{v}_1&\cdots&\boldsymbol{v}_n\end{bmatrix}\begin{pmatrix}x_1\\\vdots\\x_n\end{pmatrix}=\mathsf{V}\boldsymbol{x},\qquad\text{donde}\quad \mathsf{V}=\begin{bmatrix}\boldsymbol{v}_1&\cdots&\boldsymbol{v}_n\end{bmatrix},$$

el espacio vectorial $\mathcal{V}$ resulta ser el espacio columna de la matriz V, es decir $\mathcal{C}\left(\mathsf{V}\right)$.

Por otra parte, sabemos que sumar combinaciones lineales de columnas a otras columnas no altera el espacio columna, $\mathcal{C}\left(\mathsf{V}\right)$; y puesto que $\boldsymbol{v}_n$ es combinación lineal del resto de vectores, tenemos

$$\boldsymbol{v}_n=a_1\boldsymbol{v}_1+a_2\boldsymbol{v}_2+\cdots+a_{n-1}\boldsymbol{v}_{n-1}.$$

Sabiendo esto, podemos reducir la matriz V a una nueva matriz con la última columna compuesta por ceros (sin alterar el espacio columna) del siguiente modo:

$$\left[\frac{\mathbf{V}}{\mathbf{I}}\right] = \left[\frac{\boldsymbol{v}_1 \quad \cdots \quad \boldsymbol{v}_n}{\mathbf{I}}\right] \to \left[\begin{array}{ccccc} 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 \\ h_1 & h_2 & \dots & h_{n-1} & 0 \\ \hline & & \mathbf{E} & & \end{array}\right] = \left[\frac{\mathbf{W}}{\mathbf{E}}\right]$$

donde la última columna de **E** es $\left(-a_1, \quad -a_2, \quad \dots \quad -a_{n-1}, \quad 1,\right)$. Pero $\mathcal{V} = \mathcal{C}\left(\mathbf{V}\right) = \mathcal{C}\left(\mathbf{W}\right)$; donde el espacio columna de **W** está generado por las combinaciones lineales de las columnas no nulas de la matriz, por tanto está generado por los $n-1$ primeros vectores columna.

Si los vectores $\boldsymbol{v}_j$ pertenecen a $\mathbb{R}^m$ con $m < n$, el razonamiento es el mismo pero **W** tiene menos filas y si $m < n-1$ la matriz **W** tendrá más columnas de ceros al final.

Si los vectores $\boldsymbol{v}_j$ pertenecen a $\mathbb{R}^m$ con $m > n$, el razonamiento tampoco cambia, **W** tiene más filas, pero la última columna seguirá siendo nula.

□

(L-9) Question 15(a)

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & -2 \\ 1 & 1 & 2 & -2 \\ 2 & 2 & 2 & -4 \\ \hline 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow[{[(2)\mathbf{1}+\mathbf{4}]}]{\substack{\boldsymbol{\tau} \\ [(-1)\mathbf{1}+\mathbf{2}] \\ [(-2)\mathbf{1}+\mathbf{3}]}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 2 & 0 & -2 & 0 \\ \hline 1 & -1 & -2 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow[{[(2)\mathbf{2}+\mathbf{4}]}]{\substack{\boldsymbol{\tau} \\ [(-2)\mathbf{2}+\mathbf{3}]}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 2 & 0 & -2 & 0 \\ \hline 1 & -1 & 0 & 0 \\ 0 & 1 & -2 & 2 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right]$$

The free variable is y. The general solution is

$$\left\{ \boldsymbol{x} \in \mathbb{R}^3 \;\middle|\; \exists c \in \mathbb{R} \text{ such that } \boldsymbol{x} = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} + c \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \right\} = \left\{ \boldsymbol{x} \in \mathbb{R}^3 \;\middle|\; \exists c \in \mathbb{R} \text{ such that } \boldsymbol{x} = \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix} + c \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \right\}.$$

□

(L-9) Question 15(b) A basis is $\left[\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}; \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix};\right]$ and also $\left[\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}; \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix};\right]$.

□

(L-9) Question 16(a) **No.** El espacio P_3 tiene dimensión 4, y tan sólo hay tres vectores en este conjunto.

□

(L-9) Question 16(b) **No.** De nuevo no hay vectores suficientes para generar un espacio de dimensión 4.

□

(L-9) Question 16(c) **Si.** Estos cuatro vectores son linealmente independientes (cuatro pivotes).

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{\substack{\boldsymbol{\tau} \\ [(-1)\mathbf{2}+\mathbf{3}] \\ [(-1)\mathbf{2}+\mathbf{4}]}} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{\substack{\boldsymbol{\tau} \\ [(-1)\mathbf{3}+\mathbf{4}]}} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{\substack{\boldsymbol{\tau} \\ [\mathbf{1}\rightleftharpoons\mathbf{2}] \\ [\mathbf{2}\rightleftharpoons\mathbf{3}] \\ [\mathbf{3}\rightleftharpoons\mathbf{4}]}} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \mathbf{R}$$

□

(L-9) Question 16(d) **No.** El cuarto vector es la combinación de los tres primeros, por lo que no hay suficientes vectores linealmente independientes para generar un espacio de dimensión 4 (sólo tres pivotes)

$$\begin{bmatrix}1&1&0&0\\0&1&-1&0\\0&0&2&4\\1&2&1&4\end{bmatrix} \xrightarrow[{[(-1)\mathbf{1}+\mathbf{2}]}]{\boldsymbol{\tau}} \begin{bmatrix}1&0&0&0\\0&1&-1&0\\0&0&2&4\\1&1&1&4\end{bmatrix} \xrightarrow[{[(1)\mathbf{2}+\mathbf{3}]}]{\boldsymbol{\tau}} \begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&2&4\\1&1&2&4\end{bmatrix} \xrightarrow[{[(-2)\mathbf{3}+\mathbf{4}]}]{\boldsymbol{\tau}} \begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&2&0\\1&1&2&0\end{bmatrix}.$$

□

(L-9) Question 17(a) Si $\boldsymbol{u}_1$ y $\boldsymbol{u}_2$ son linealmente dependientes, entonces existe un número a tal que

$$\boldsymbol{u}_1 = a\boldsymbol{u}_2;$$

es decir, que componente a componente $u_{1i} = a \cdot u_{2i}$. Puesto que las terceras componentes son iguales, dicho número debería ser $a = 1$, pero entonces la primera componente de $\boldsymbol{u}_1$ también debería ser una vez la primera componente de $\boldsymbol{u}_2$. Puesto que no es así, el primer vector no es un múltiplo del segundo. Así pues no existe tal número a y, por tanto, estos vectores son linealmente *independientes*.

Otra forma de verlo es comprobar que el rango de la matriz $\mathbf{M} = \begin{bmatrix}\boldsymbol{u}_1 & \boldsymbol{u}_2\end{bmatrix}$ es dos.

□

(L-9) Question 17(b) La respuesta es sí. Si escribiéramos los vectores en forma de columna, esta pregunta sería equivalente a preguntar si $\boldsymbol{v}$ (en forma de columna) pertenece al espacio columna de la matriz

$$\mathbf{A} = \begin{bmatrix}1&1\\0&-1\\1&1\end{bmatrix};$$

es decir, a preguntar si el sistema

$$\mathbf{A}\boldsymbol{x} = \begin{bmatrix}1&1\\0&-1\\1&1\end{bmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix} = \begin{pmatrix}2\\1\\2\end{pmatrix}$$

tiene solución. Puesto que 3 veces la primera columna menos la segunda da el resultado deseado, $\boldsymbol{v} = (2, 1, 2)$ pertenece al espacio generado por $\{\boldsymbol{u}_1, \boldsymbol{u}_2\}$.

Otra forma de verlo es comprobar que la matriz

$$\mathbf{N} = \begin{bmatrix}\boldsymbol{u}_1 & \boldsymbol{u}_2 & \boldsymbol{v}\end{bmatrix}$$

también tiene rango 2, es decir, que al añadir el vector $\boldsymbol{v}$, el rango de la nueva matriz sigue siendo 2 (como el rango de la matriz $\mathbf{M}$ del primer apartado).

□

(L-9) Question 17(c) Debemos encontrar un tercer vector linealmente independiente de $\boldsymbol{u}_1$ y $\boldsymbol{u}_2$. Si escribimos de nuevo los vectores en columna, buscamos un vector $\boldsymbol{b}$ que no pertenezca al espacio columna de $\mathbf{A}$. Por tanto queremos un $\boldsymbol{b}$ tal que $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$ no tenga solución.

$$\left[\begin{array}{c|c}\mathbf{A} & \text{-}\boldsymbol{b}\\ \hline \mathbf{I} & \mathbf{0}\end{array}\right] = \left[\begin{array}{cc|c}1&0&-b_1\\0&-1&-b_2\\1&0&-b_3\\ \hline 1&0&0\\0&1&0\\ \hline 0&0&1\end{array}\right] \xrightarrow[{\substack{[(b_1)\mathbf{1}+\mathbf{3}]\\ [(-b_2)\mathbf{2}+\mathbf{3}]}}]{\boldsymbol{\tau}} \left[\begin{array}{cc|c}1&0&0\\0&-1&0\\1&0&b_1-b_3\\ \hline 1&0&b1\\0&1&-b2\\ \hline 0&0&1\end{array}\right]$$

Por tanto, si $b_1 \neq b_3$ el sistema no tiene solución, es decir, $\boldsymbol{b}$ no es combinación lineal de los otros dos, y por tanto tenemos tres vectores de $\mathbb{R}^3$ linealmente independientes, es decir, una base de $\mathbb{R}^3$.

□

(L-9) Question 18(a) Si. Dos vectores son dependientes si uno es un múltiplo del otro; pero en este caso no es así. Por tanto son linealmente independientes.

□

(L-9) Question 18(b) No, los vectores $\boldsymbol{v}_1$, $\boldsymbol{v}_3$, $\boldsymbol{v}_3$ y $\boldsymbol{v}_4$ no son linealmente independientes. Por ejemplo $\boldsymbol{v}_1 = \boldsymbol{v}_4$.

□

(L-9) Question 18(c) No, los vectores no son base de dicho sub-espacio.

Los vectores son linealmente independientes, pero no generan el plano descrito en el enunciado ya que $\boldsymbol{v}_3$ no está en dicho plano (no satisface la ecuación $x_1 + 2x_2 + 3x_3 + 6x_4 = 0$).

□

(L-9) Question 18(d) Lo resolveremos por eliminación

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & -q \\ 4 & 2 & 12 & -3 \\ 6 & 2 & 10 & -1 \\ \hline 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{\substack{\boldsymbol{\tau} \\ {[(1)\mathbf{1}+\mathbf{3}]} \\ {[(q)\mathbf{1}+\mathbf{4}]}}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 4 & 2 & 16 & 4q-3 \\ 6 & 2 & 16 & 6q-1 \\ \hline 1 & 0 & 1 & q \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{\substack{\boldsymbol{\tau} \\ {[(-8)\mathbf{2}+\mathbf{3}]} \\ {[(2)\mathbf{4}]} \\ {[(-4q+3)\mathbf{2}+\mathbf{4}]}}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 4 & 2 & 0 & 0 \\ 6 & 2 & 0 & 4q+4 \\ \hline 1 & 0 & 1 & 2q \\ 0 & 1 & -8 & -4q+3 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 2 \end{array}\right]$$

Los cuatro vectores generan todo $\mathbb{R}^3$ si $q \neq -1$ (tres pivotes). Pero si $q = -1$, los cuatro vectores solo generan un subespacio de dimensión $r = 2$ (dos pivotes), es decir, generan un plano en $\mathbb{R}^3$ que pasa por el origen.

□

(L-9) Question 19(a)

$$\boldsymbol{u} = \begin{pmatrix}1\\0\\0\end{pmatrix}, \; \boldsymbol{v} = \begin{pmatrix}2\\0\\0\end{pmatrix}, \; \boldsymbol{w} = \begin{pmatrix}3\\0\\0\end{pmatrix}, \; \boldsymbol{z} = \begin{pmatrix}0\\1\\0\end{pmatrix}; \qquad \mathbf{A} = \begin{bmatrix}-1 & -1\\ 0 & 0\\ 0 & 0\end{bmatrix}.$$

□

(L-9) Question 19(b) Son el número de pivotes y el número de columnas libres respectivamente:

$$\dim(\mathcal{C}(\mathbf{A})) = 1; \qquad \dim(\mathcal{N}(\mathbf{A})) = 1.$$

□

(L-10) Question 1(a)

$$\left[\begin{array}{c}\mathbf{A}\\ \hline \mathbf{I}\end{array}\right] = \left[\begin{array}{cccc} 0 & 1 & 4 & 0 \\ 0 & 2 & 8 & 0 \\ \hline 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{\substack{\boldsymbol{\tau} \\ {[(-4)\mathbf{2}+\mathbf{3}]}}} \left[\begin{array}{cccc} 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ \hline 1 & 0 & 0 & 0 \\ 0 & 1 & -4 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right] = \left[\begin{array}{c}\mathbf{K}\\ \hline \mathbf{E}\end{array}\right]$$

From its pre-echelon form $\mathbf{K}$ we known that only the second one is a pivot column, therefore

$$\mathcal{C}(\mathbf{A}) = \left\{ \boldsymbol{x} \in \mathbb{R}^2 \;\middle|\; \exists c \in \mathbb{R} \text{ such that } \boldsymbol{x} = c\begin{pmatrix}1\\2\end{pmatrix} \right\}; \quad \boxed{\dim \mathcal{C}(\mathbf{A}) = 1. \quad \text{Basis for } \mathcal{C}(\mathbf{A}) \text{: } \left[\begin{pmatrix}1\\2\end{pmatrix};\right].}$$

From the columns of $\mathbf{E}$ below the null columns of $\mathbf{K}$ we see that $\boxed{\dim \mathcal{N}(\mathbf{A}) = 3}$ and

$$\mathcal{N}(\mathbf{A}) = \left\{ \boldsymbol{x} \in \mathbb{R}^4 \;\middle|\; \exists \boldsymbol{a} \in \mathbb{R}^3 \text{ such that } \boldsymbol{x} = \begin{bmatrix}1 & 0 & 0\\ 0 & -4 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\boldsymbol{a} \right\}. \quad \boxed{\text{Basis for } \mathcal{N}(\mathbf{A}) \text{: } \left[\begin{pmatrix}1\\0\\0\\0\end{pmatrix}; \begin{pmatrix}0\\-4\\1\\0\end{pmatrix}; \begin{pmatrix}0\\0\\0\\1\end{pmatrix};\right].}$$

Since there is only one pivot row $\boxed{\dim \mathcal{C}(\mathbf{A}^\intercal) = 1}$ and

$$\mathcal{C}(\mathbf{A}^\intercal) = \left\{ \boldsymbol{x} \in \mathbb{R}^4 \mid \exists c \in \mathbb{R} \text{ such that } \boldsymbol{x} = c\,(0, \;\; 1, \;\; 4, \;\; 0,) \right\}. \quad \boxed{\text{Basis for } \mathcal{C}(\mathbf{A}^\intercal) \text{: } \left[(0,\ 1,\ 4,\ 0,);\right].}$$

Since

$$\begin{bmatrix}1&0\\-2&1\end{bmatrix}\mathbf{K}=\begin{bmatrix}1&0\\-2&1\end{bmatrix}\begin{bmatrix}0&1&0&0\\0&2&0&0\end{bmatrix}=\begin{bmatrix}0&1&0&0\\0&0&0&0\end{bmatrix}$$

$$\mathcal{N}\left(\mathbf{A}^{\intercal}\right)=\left\{\boldsymbol{x}\in\mathbb{R}^2\;\middle|\;\exists\, c\in\mathbb{R}\text{ such that }\boldsymbol{x}=c\left(-2,\quad 1,\right)\right\};\quad \boxed{\dim\mathcal{N}\left(\mathbf{A}^{\intercal}\right)=1.\quad\text{Basis for }\mathcal{N}\left(\mathbf{A}^{\intercal}\right)\text{: }\left[(-2,\ 1,);\right].}$$

□

(L-10) Question 1(b) $\dim\mathcal{C}\left(\mathbf{A}\right)+\dim\mathcal{N}\left(\mathbf{A}^{\intercal}\right)=1+1=2=m.$
$\dim\mathcal{C}\left(\mathbf{A}^{\intercal}\right)+\dim\mathcal{N}\left(\mathbf{A}\right)=1+3=4=n$

□

(L-10) Question 1(c)

$$\left[\begin{array}{cccc}0&1&4&0\\0&0&0&0\\\hline 1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{array}\right]\xrightarrow[{[(-4)\mathbf{2}+\mathbf{3}]}]{\boldsymbol{\tau}}\left[\begin{array}{cccc}0&1&0&0\\0&0&0&0\\\hline 1&0&0&0\\0&1&-4&0\\0&0&1&0\\0&0&0&1\end{array}\right]$$

Since only the second is a pivot column, then

$$\mathcal{C}\left(\mathbf{U}\right)\;=\;\mathcal{C}\left(\mathbf{A}\right)\;=\;\left\{\boldsymbol{x}\in\mathbb{R}^2\;\middle|\;\exists\, c\in\mathbb{R}\text{ such that }\boldsymbol{x}=c\begin{pmatrix}1\\0\end{pmatrix}\right\};\quad\boxed{\dim\mathcal{C}\left(\mathbf{A}\right)=1.\quad\text{Basis for }\mathcal{C}\left(\mathbf{A}\right)\text{: }\left[\begin{pmatrix}1\\0\end{pmatrix};\right].}$$

$\mathcal{N}\left(\mathbf{U}\right)=\mathcal{N}\left(\mathbf{A}\right)$:

$$\mathcal{N}\left(\mathbf{U}\right)=\left\{\boldsymbol{x}\in\mathbb{R}^4\;\middle|\;\exists\,\boldsymbol{a}\in\mathbb{R}^3\text{ such that }\boldsymbol{x}=\begin{bmatrix}1&0&0\\0&-4&0\\0&1&0\\0&0&1\end{bmatrix}\boldsymbol{a}\right\}.\quad\boxed{\text{Basis for }\mathcal{N}\left(\mathbf{A}\right)\text{: }\left[\begin{pmatrix}1\\0\\0\\0\end{pmatrix};\begin{pmatrix}0\\-4\\1\\0\end{pmatrix};\begin{pmatrix}0\\0\\0\\1\end{pmatrix};\right].}$$

$\mathcal{C}\left(\mathbf{U}^{\intercal}\right)=\mathcal{C}\left(\mathbf{A}^{\intercal}\right)$:

$$\mathcal{C}\left(\mathbf{U}^{\intercal}\right)\quad=\quad\left\{\boldsymbol{x}\in\mathbb{R}^4\;\middle|\;\exists\, c\in\mathbb{R}\text{ such that }\boldsymbol{x}=c\left(0,\quad 1,\quad 4,\quad 0,\right)\right\}.\quad\boxed{\text{Basis for }\mathcal{C}\left(\mathbf{A}^{\intercal}\right)\text{: }\left[(0,\ 1,\ 4,\ 0,);\right].}$$

Since the only free row is the second one: "a zero vector"

$$\mathcal{N}\left(\mathbf{U}^{\intercal}\right)\;=\;\left\{\boldsymbol{x}\in\mathbb{R}^2\;\middle|\;\exists\, c\in\mathbb{R}\text{ such that }\boldsymbol{x}=c\left(0,\quad 1,\right)\right\};\quad\boxed{\dim\mathcal{N}\left(\mathbf{A}^{\intercal}\right)=1.\quad\text{Basis for }\mathcal{N}\left(\mathbf{A}^{\intercal}\right)\text{: }\left[(0,\ 1,);\right].}$$

□

(L-10) Question 1(d) $\dim\mathcal{C}\left(\mathbf{U}\right)+\dim\mathcal{N}\left(\mathbf{U}^{\intercal}\right)=1+1=2=m.$
$\dim\mathcal{C}\left(\mathbf{U}^{\intercal}\right)+\dim\mathcal{N}\left(\mathbf{U}\right)=1+3=4=n$

□

(L-10) Question 2. The column space $\mathcal{C}\left(\mathbf{A}\right)$ is the set of linear combinations of the two last columns

$$\left\{\boldsymbol{x}\in\mathbb{R}^3\;\middle|\;\exists\boldsymbol{a}\in\mathbb{R}^3\text{ such that }\boldsymbol{x}=\begin{bmatrix}1&0\\0&1\\0&0\end{bmatrix}\boldsymbol{a}\right\}$$

that is, all vectors in $\mathbb{R}^3$ with a zero as a third component (a subspace of dimension 2).

The null space $\mathcal{N}\left(\mathbf{A}\right)$ is the set of all multiples

$$\left\{ \boldsymbol{x} \in \mathbb{R}^3 \;\middle|\; \exists c \in \mathbb{R} \text{ such that } \boldsymbol{x} = c \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\}$$

that is, all vectors in $\mathbb{R}^3$ with zero as second and third components (a subspace of dimension 1).

The row space $\mathcal{C}\left(\mathbf{A}^\mathsf{T}\right)$ is the set of all the vectors $\boldsymbol{x}$ such that

$$\left\{ \boldsymbol{x} \in \mathbb{R}^3 \;\middle|\; \exists \boldsymbol{a} \in \mathbb{R}^2 \text{ such that } \boldsymbol{x} = \boldsymbol{a} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\}$$

or the set of all vectors in $\mathbb{R}^3$ with zero as a first component (a subspace of dimension 1).

$\mathcal{N}\left(\mathbf{A}^\mathsf{T}\right)$ is the set of all the multiples

$$\left\{ \boldsymbol{x} \in \mathbb{R}^3 \;\middle|\; \exists c \in \mathbb{R} \text{ such that } \boldsymbol{x} = c \left(0, \quad 0, \quad 1,\right) \right\}$$

that is, all vectors in $\mathbb{R}^3$ with zero as first and second components (a subspace of dimension 1).

□

(L-10) Question 3(a) Row and column space dimension = 5; nullspace dimension = 4; left nullspace dimension = 2; sum = 16 = $m + n$.

□

(L-10) Question 3(b) $\mathcal{C}\left(\mathbf{A}\right) = \mathbb{R}^3$; $\quad \mathcal{N}\left(\mathbf{A}^\mathsf{T}\right) = \{\mathbf{0}\}$.

□

(L-10) Question 4. **No.** Consider any two invetible matrices n by n; both have the same four subspaces.

□

(L-10) Question 5. Since $\mathbf{U}$ has two pivots, the rank is two for both matrices.

The row space for both matrices is the same (column operations do not change the column space), but $\mathcal{C}\left(\mathbf{A}^\mathsf{T}\right)$ is different from $\mathcal{C}\left(\mathbf{L}^\mathsf{T}\right)$. Note that all the vectors in $\mathcal{C}\left(\mathbf{L}^\mathsf{T}\right)$ have the third and fourth components equal to zero.

$$\mathcal{C}\left(\mathbf{A}\right) = \mathcal{C}\left(\mathbf{L}\right) = \left\{ \boldsymbol{x} \in \mathbb{R}^3 \;\middle|\; \text{exists } a, b \in \mathbb{R} \text{ such that } \boldsymbol{x} = a \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + b \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\} \quad \boxed{\text{Basis for } \mathcal{C}\left(\mathbf{A}\right) \text{ and } \mathcal{C}\left(\mathbf{L}\right)\text{: } \left[\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} ; \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} ; \right]}$$

$$\mathcal{C}\left(\mathbf{A}^\mathsf{T}\right) = \left\{ \boldsymbol{x} \in \mathbb{R}^4 \;\middle|\; \text{exists } a, b \in \mathbb{R} \text{ such that } \boldsymbol{x} = a \begin{pmatrix} 1 \\ 2 \\ 0 \\ 1 \end{pmatrix} + b \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} \right\} \quad \boxed{\text{Basis for } \mathcal{C}\left(\mathbf{A}^\mathsf{T}\right)\text{: } \left[\begin{pmatrix} 1 \\ 2 \\ 0 \\ 1 \end{pmatrix} ; \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} ; \right].}$$

$$\mathcal{C}\left(\mathbf{L}^\mathsf{T}\right) = \left\{ \boldsymbol{x} \in \mathbb{R}^4 \;\middle|\; \text{exists } a, b \in \mathbb{R} \text{ such that } \boldsymbol{x} = a \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + b \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \right\} \quad \boxed{\text{Basis for } \mathcal{C}\left(\mathbf{L}^\mathsf{T}\right)\text{: } \left[\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} ; \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} ; \right].}$$

Since there are two free columns:

$$\mathcal{N}\left(\mathbf{A}\right) = \left\{ \boldsymbol{x} \in \mathbb{R}^4 \;\middle|\; \text{exists } a, b \in \mathbb{R} \text{ such that } \boldsymbol{x} = a \begin{pmatrix} 2 \\ -1 \\ 1 \\ 0 \end{pmatrix} + b \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\} \quad \boxed{\text{Basis for } \mathcal{N}\left(\mathbf{A}\right)\text{: } \left[\begin{pmatrix} 2 \\ -1 \\ 1 \\ 0 \end{pmatrix} ; \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix} ; \right].}$$

$$\mathcal{N}\left(\mathbf{L}\right) = \left\{ \boldsymbol{x} \in \mathbb{R}^4 \middle| \text{exists } a, b \in \mathbb{R} \text{ such that } \boldsymbol{x} = a\begin{pmatrix}0\\0\\1\\0\end{pmatrix} + b\begin{pmatrix}0\\0\\0\\1\end{pmatrix} \right\} \qquad \boxed{\text{Basis for } \mathcal{N}\left(\mathbf{A}^\intercal\right)\text{: } \left[\begin{pmatrix}0\\0\\1\\0\end{pmatrix};\begin{pmatrix}0\\0\\0\\1\end{pmatrix};\right].}$$

There is only one free row and therefore the dimension of the left null space is one for both matrices; and the left null space is for both matrices.

$$\mathcal{N}\left(\mathbf{A}^\intercal\right) = \mathcal{N}\left(\mathbf{L}^\intercal\right) = \left\{ \boldsymbol{x} \in \mathbb{R}^3 \middle| \text{exists } a \in \mathbb{R} \text{ such that } \boldsymbol{x} = a\begin{pmatrix}1, & 0, & -1,\end{pmatrix} \right\} \qquad \boxed{\text{Basis for } \mathcal{N}\left(\mathbf{A}^\intercal\right) \text{ and } \mathcal{N}\left(\mathbf{L}^\intercal\right)\text{: } \left[\begin{pmatrix}1\\0\\-1\end{pmatrix};\right].}$$

□

(L-10) Question 6(a) Dimension 3:

$$\mathcal{V} = \left\{ \boldsymbol{x} \in \mathbb{R}^4 \middle| \; x_1 + x_2 + x_3 + x_4 = 0 \right\}$$

therefore

$$\mathcal{V} = \mathcal{N}\left(\mathbf{A}\right); \quad \text{where} \quad \mathbf{A} = \begin{bmatrix}1 & 1 & 1 & 1\end{bmatrix}.$$

Hence

$$\left[\begin{array}{cccc} 1 & 1 & 1 & 1 \\ \hline 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow[{[(-1)\mathbf{1}+\mathbf{4}]}]{\substack{\boldsymbol{\tau} \\ [(-1)\mathbf{1}+\mathbf{2}] \\ [(-1)\mathbf{1}+\mathbf{3}]}} \left[\begin{array}{cccc} 1 & 0 & 0 & 0 \\ \hline 1 & -1 & -1 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

□

(L-10) Question 6(b) Since the only vector en $\mathcal{N}\left(\mathbf{I}\right)$ is the zero vector $\mathbf{0}$, the dimension is 0.

□

(L-10) Question 6(c) Dimension 16.

□

(L-10) Question 7. A basis of the row space is

$$\left[\begin{pmatrix}3\\0\\3\end{pmatrix};\begin{pmatrix}1\\1\\2\end{pmatrix};\right].$$

A basis of the column space is

$$\left[\begin{pmatrix}1\\4\\2\end{pmatrix};\begin{pmatrix}2\\5\\7\end{pmatrix};\right].$$

Because the rank is two.

□

(L-10) Question 8(a) No. This is not a vector space because $\mathbf{0}$ is not in this subspace.

□

(L-10) Question 8(b) Yes. (This is actually just the left nullspace of the matrix whose columns are $\boldsymbol{y}$ and $\boldsymbol{z}$.)

$$\boldsymbol{x}\begin{bmatrix}\boldsymbol{z} & \boldsymbol{y}\end{bmatrix} = \mathbf{0}.$$

□

(L-10) Question 8(c) No. For example, the zero matrix $\mathbf{0}$ is not in this subset.

□

(L-10) Question 8(d) Yes. If the nullspaces of $\mathbf{A}_1$ and $\mathbf{A}_2$ contain $\begin{pmatrix}1\\2\\3\end{pmatrix}$ then any linear combination of these matrices does too:

$$(a\mathbf{A}_1 + b\mathbf{A}_1)\begin{pmatrix}1\\2\\3\end{pmatrix} = a\mathbf{A}_1\begin{pmatrix}1\\2\\3\end{pmatrix} + b\mathbf{A}_2\begin{pmatrix}1\\2\\3\end{pmatrix} = a\mathbf{0} + b\mathbf{0} = \mathbf{0}.$$

□

(L-10) Question 9. Si empleamos el método de eliminación gaussiana, encontramos que sólo hay dos pivotes, que corresponden con las dos primeras columnas y las dos primeras filas. Así que sabemos que tanto el espacio columna y el espacio fila tienen dimensión dos. Por tanto, tomemos las dos primeras columnas y las dos primeras filas, puesto que son linealmente independientes.

$$\mathcal{C}\left(\mathbf{A}\right) = \left\{ \boldsymbol{x} \in \mathbb{R}^3 \;\middle|\; \exists\, c, d \in \mathbb{R} \text{ such that } \boldsymbol{x} = c\begin{pmatrix}1\\2\\1\end{pmatrix} + d\begin{pmatrix}3\\-1\\4\end{pmatrix} \right\}$$
$$\mathcal{C}\left(\mathbf{A}^\intercal\right) = \left\{ \boldsymbol{x} \in \mathbb{R}^4 \;\middle|\; \exists\, c, d \in \mathbb{R} \text{ such that } \boldsymbol{x} = c\begin{pmatrix}1 & 3 & 5 & -2\end{pmatrix} + d\begin{pmatrix}2 & -1 & 3 & -4\end{pmatrix} \right\}.$$

□

(L-10) Question 10(a) $\mathcal{N}\left(\mathbf{A}\right) = \mathbf{0}$.

□

(L-10) Question 10(b) $\dim \mathcal{N}\left(\mathbf{A}^\intercal\right) = 1$.

□

(L-10) Question 10(c) $\boldsymbol{x}_p = \begin{pmatrix}0\\1\\0\\0\end{pmatrix}$.

□

(L-10) Question 10(d) $\boldsymbol{x} = \boldsymbol{x}_p + \mathbf{0} = \boldsymbol{x}_p = \begin{pmatrix}0\\1\\0\\0\end{pmatrix}$.

□

(L-10) Question 10(e)

$$\mathbf{R} = \begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\\a&b&c&d\end{bmatrix},$$

where the last row of $\mathbf{R}$ is not possible to knowm without more information.

□

(L-10) Question 11(a) Falso. Por ejemplo para la matriz $\mathbf{A} = \begin{bmatrix}0&1&0\\0&0&1\\0&0&0\end{bmatrix}$; $\mathcal{C}\left(\mathbf{A}\right)$ es el sub-espacio de vectores de $\mathbb{R}^3$ con la última componente igual a cero; mientras que $\mathcal{C}\left(\mathbf{A}^\intercal\right)$ es el sub-espacio de vectores de $\mathbb{R}^3$ con la primera componente nula.

□

(L-10) Question 11(b) Verdadero.

□

(L-10) Question 11(c) Falso. Suponga dos matrices invertibles, por ejemplo

$$\mathbf{I} = \begin{bmatrix}1&0\\0&1\end{bmatrix} \quad \text{y} \quad \mathbf{A} = \begin{bmatrix}1&0\\0&2\end{bmatrix}.$$

$$\mathcal{C}\left(\mathbf{A}\right) = \mathcal{C}\left(\mathbf{I}\right) = \mathcal{C}\left(\mathbf{A}^\intercal\right) = \mathcal{C}\left(\mathbf{I}^\intercal\right) = \mathbb{R}^2 \quad \text{y} \quad \mathcal{N}\left(\mathbf{A}\right) = \mathcal{N}\left(\mathbf{I}\right) = \mathcal{N}\left(\mathbf{A}^\intercal\right) = \mathcal{N}\left(\mathbf{I}^\intercal\right) = \{\mathbf{0}\}.$$

□

(L-10) Question 11(d) Verdadero. Sea cual sea el vector $\boldsymbol{b}$, las n columnas (vectores de $\mathbb{R}^n$) son linealmente independientes, por tanto son una base de $\mathbb{R}^n$, puesto que el lado derecho $\boldsymbol{b}$ pertenece a $\mathbb{R}^n$, siempre existe una única combinación lineal de las columnas igual a $\boldsymbol{b}$ (por ser estas una base de $\mathbb{R}^n$). Dicha combinación es *"la"* solución $\boldsymbol{x}$ al sistema de ecuaciones.

Otra forma de verlo es la siguiente: si las n columnas son linealmente independientes, $\mathbf{A}$ es de rango completo y por lo tanto invertible; entonces

$$\begin{aligned} \mathbf{A}\boldsymbol{x} =& \boldsymbol{b} \\ \mathbf{A}^{-1}\mathbf{A}\boldsymbol{x} =& \mathbf{A}^{-1}\boldsymbol{b} \\ \boldsymbol{x} =& \mathbf{A}^{-1}\boldsymbol{b} \end{aligned}$$

Por tanto sabemos que para cualquier $\boldsymbol{b}$, el vector $\mathbf{A}^{-1}\,\boldsymbol{b}$ es la solución.

□

(L-10) Question 12(a) Puesto que cualquier combinación del espacio columna $\mathbf{A}\boldsymbol{x}$ es un múltiplo de $\begin{pmatrix}-2, & 1,\end{pmatrix}$, dicho espacio es un subespacio de $\mathbb{R}^2$ de dimensión 1 (una recta en $\mathbb{R}^2$); por tanto el rango de $\mathbf{A}$ es $r = 1$ (una sola columna pivote, y consecuentemente sólo una fila pivote).

Además, sabemos que las dimensiones de $\mathbf{A}$ son $m = 2$ y $n = 4$ es decir: $\underset{2\times 4}{\mathbf{A}}$.

□

(L-10) Question 12(b) Número de columnas libres, es decir, el número de columnas menos el número de columnas pivote: $\dim \mathcal{N}\left(\mathbf{A}\right) = 4 - r = 4 - 1 = 3$.

□

(L-10) Question 12(c) $\dim \mathcal{C}\left(\mathbf{A}^\intercal\right) = r = 1$.

□

(L-10) Question 12(d) El número de filas libres; por tanto $m - r = 2 - 1 = 1$.

□

(L-10) Question 12(e) Sabemos que

$$\mathbf{A}\boldsymbol{v} = \mathbf{A}\begin{pmatrix}1\\-2\\3\\1\end{pmatrix} = \begin{pmatrix}-6\\3\end{pmatrix} \qquad \text{y que} \qquad \mathbf{A}\boldsymbol{w} = \mathbf{A}\begin{pmatrix}3\\-1\\1\\2\end{pmatrix} = \begin{pmatrix}-18\\9\end{pmatrix}.$$

Por tanto,

$$-3\mathbf{A}\boldsymbol{v} + \mathbf{A}\boldsymbol{w} = \begin{pmatrix}18\\-9\end{pmatrix} + \begin{pmatrix}-18\\9\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}.$$

Es decir

$$-3\mathbf{A}\boldsymbol{v} + \mathbf{A}\boldsymbol{w} = \mathbf{A}(-3\boldsymbol{v}) + \mathbf{A}\boldsymbol{w} = \mathbf{A}(-3\boldsymbol{v} + \boldsymbol{w}) = \mathbf{0}.$$

Así pues, una solución al sistema homogéneo es:

$$\boldsymbol{x} = -3\boldsymbol{v} + \boldsymbol{w} = \begin{pmatrix}-3\\6\\-9\\-3\end{pmatrix} + \begin{pmatrix}3\\-1\\1\\2\end{pmatrix} = \begin{pmatrix}0\\5\\-8\\-1\end{pmatrix}.$$

También es solución cualquier múltiplo de dicho vector $\boldsymbol{x}$.

□

(L-10) Question 13(a)

$\operatorname{rg}\left(\mathbf{A}\right) =$ number of pivots $= 3$.

$\dim \mathcal{C}\left(\mathbf{A}\right) = \dim \mathcal{C}\left(\mathbf{A}^\intercal\right) = \operatorname{rg}\left(\mathbf{A}\right) = 3$.

$\dim \mathcal{N}\left(\mathbf{A}\right) =$ number of free columns $= 5 - 3 = 2$.

□

(L-10) Question 13(b) The rows 1, 2 and 4 of $\mathbf{A}$ (the three pivot rows of $\mathbf{A}$).

□

(L-10) Question 13(c) The pivot columns of $\mathbf{R}$ (also the pivot columns of $\mathbf{A}$).

□

(L-10) Question 13(d) The columns 3 and 4 of $\mathbf{E}$.

□

(L-10) Question 13(e) $3\mathbf{A}_{|1} - 2\mathbf{A}_{|2} = \mathbf{A}_{|3}$. Note that the third column of $\mathbf{E}$ is telling us a linear combination. We can find more, but that is the only possibility if we only use the first and second columns.

□

(L-10) Question 14. Since both solutions belong to $\mathbb{R}^4$, we know $\mathbf{A}$ has 4 colunms. Since the whole nullspace consists of all linear combination of only two vectors, only two of the columns are free, and the other two are pivots (rank 2). Hence, the dimension of row space is two, and it is orthogonal to the given vectors.

Lets compute one possible answer using gaussian elimination.

$$\left[\begin{array}{cccc} 2 & 2 & 1 & 0 \\ 3 & 1 & 0 & 1 \\ \hline 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow[{[\mathbf{1}\rightleftharpoons\mathbf{3}]}]{\tau} \left[\begin{array}{cccc} 1 & 2 & 2 & 0 \\ 0 & 1 & 3 & 1 \\ \hline 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow[\substack{[(-2)\mathbf{1}+\mathbf{2}] \\ [(-2)\mathbf{1}+\mathbf{3}] \\ [(-3)\mathbf{2}+\mathbf{3}] \\ [(-1)\mathbf{2}+\mathbf{4}]}]{\tau} \left[\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & \mathbf{1} & \mathbf{0} \\ 0 & 1 & \mathbf{-3} & \mathbf{-1} \\ 1 & -2 & \mathbf{4} & \mathbf{2} \\ 0 & 0 & \mathbf{0} & \mathbf{1} \end{array}\right]$$

Now we've got a basis for the row space. Thus, a possible answer is

$$\mathbf{A} = \begin{bmatrix} 1 & -3 & 4 & 0 \\ 0 & -1 & 2 & 1 \end{bmatrix}.$$

It's easy to check...

$$\begin{bmatrix} 1 & -3 & 4 & 0 \\ 0 & -1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 2 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

But we can find more matrices as far as we multiply $\mathbf{A}$ on the left, and preserve the rank (preserve the number of free variables). Thus, any matrix obtained by row elementary operations (row operations without changing the rank) on $\mathbf{A}$ is a valid one

$$\underset{2\times 2}{\mathbf{E}}\;\underset{2\times 4}{\mathbf{A}} \begin{bmatrix} 2 & 3 \\ 2 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} = \mathbf{E0} = \mathbf{0};$$

where $\operatorname{rg}\big(\mathbf{EA}\big) = 2$.

□

(L-10) Question 15. By column gaussian elimination from right to left we get

$$\left[\begin{array}{cccc} 4 & 3 & 2 & 1 \\ \hline 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow[\substack{[(-2)\mathbf{4}+\mathbf{3}] \\ [(-3)\mathbf{4}+\mathbf{2}] \\ [(-4)\mathbf{4}+\mathbf{1}]}]{\tau} \left[\begin{array}{cccc} 0 & 0 & 0 & 1 \\ \hline \mathbf{1} & \mathbf{0} & \mathbf{0} & 0 \\ \mathbf{0} & \mathbf{1} & \mathbf{0} & 0 \\ \mathbf{0} & \mathbf{0} & \mathbf{1} & 0 \\ \mathbf{-4} & \mathbf{-3} & \mathbf{-2} & 1 \end{array}\right]$$

Then:

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 & -4 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & -2 \end{bmatrix}$$

But any matrix obtained by row elementary operations (row operations without changing the rank) on $\mathbf{A}$ is a valid one.

□

(L-10) Question 16(a) Suponga que el producto de $\mathbf{A}$ y $\mathbf{B}$ es la matriz nula: $\mathbf{AB} = \mathbf{0}$. Entonces el espacio *nulo* de la matriz $\mathbf{A}$ contiene el espacio *columna* de la matriz $\mathbf{B}$. También el espacio *nulo por la izquierda* de la matriz $\mathbf{B}$ contiene el espacio *fila* de la matriz $\mathbf{A}$.

□

(L-10) Question 16(b) La dimensión del espacio nulo de $\mathbf{A}$ es $n - r = 7 - r$. La dimensión del espacio columna de $\mathbf{B}$ es s. Puesto que el primero contiene al segundo, $7 - r \geq s$, es decir $r + s \leq 7$.

□

(L-10) Question 17(a) 4. There are 4 pivots in the reduced echelon form of $\mathbf{A}$.

□

(L-10) Question 17(b)
$\dim \mathcal{C}\left(\mathbf{A}\right) = \dim \mathcal{C}\left(\mathbf{A}^\intercal\right) = \operatorname{rg}\left(A\right) = 4$
$\dim \mathcal{N}\left(\mathbf{A}\right) = n - \operatorname{rg}\left(A\right) = 8 - 4 = 4$
$\dim \mathcal{N}\left(\mathbf{A}^\intercal\right) = m - \operatorname{rg}\left(A\right) = 4 - 4 = 0$

□

(L-10) Question 17(c) $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$ will have infinitely many solutions for any $\boldsymbol{b}$. There is no row of 0's in the reduced column echelon form to cause there to be no solutions for the "wrong" $\boldsymbol{b}$. There are infinitely many solutions since the nullspace, being 4-dimensional, has infinitely many elements.

□

(L-10) Question 17(d) Yes. The reduced column echelon form of $\mathbf{A}$ has linearly independent rows.

□

(L-10) Question 17(e) Columns 2, 4, 5 and 7 of $\mathbf{E}$.

□

(L-10) Question 17(f) We saw that $\dim(\mathcal{N}\left(\mathbf{A}^\intercal\right)) = 0$. Hence, $\mathcal{N}\left(\mathbf{A}^\intercal\right)$ contains only the zero vector; and the isn't any basis for this space.

□

(L-10) Question 17(g) It is imposible, it is a singular matrix.

□

(L-10) Question 17(h) $\left[\mathbf{E}_{|1};\ \mathbf{E}_{|3};\ \mathbf{E}_{|6};\ \mathbf{E}_{|8};\right]$.

□

(L-10) Question 18(a) Since the three columns are pivot colums (no free columns), then $\mathcal{N}\left(\mathbf{R}\right) = \{\mathbf{0}\}$.

$$\mathcal{N}\left(\mathbf{R}\right) = \left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right\}$$

□

(L-10) Question 18(b) $\mathbf{B}$ is its column reduced echelon form. Therefore, the rank is 3.

□

(L-10) Question 18(c)

$$\begin{bmatrix} \mathbf{R} & \mathbf{R} \\ \mathbf{R} & \mathbf{0} \end{bmatrix} \xrightarrow[\text{from the 5 last ones}]{\text{substracting the five first columns.}} \begin{bmatrix} \mathbf{R} & \mathbf{0} \\ \mathbf{R} & -\mathbf{R} \end{bmatrix} \xrightarrow[\text{to the first ones}]{\text{adding the last ones.}} \begin{bmatrix} \mathbf{R} & \mathbf{0} \\ \mathbf{0} & -\mathbf{R} \end{bmatrix} \xrightarrow[\text{columns by -1}]{\text{multiplying the last.}} \begin{bmatrix} \mathbf{R} & \mathbf{0} \\ \mathbf{0} & \mathbf{R} \end{bmatrix}$$

□

(L-10) Question 18(d) Twice the rank of $\mathbf{R}$, that is, rank 6.

□

(L-10) Question 18(e) Must be equal to the number of zero rows of $\underset{10\times 6}{\mathbf{C}}$ (the zero columns of $\mathbf{C}$; since the six columns of $\mathbf{C}$ are pivot columns (rank 6); only 6 rows are pivot rows, the remaining are free; therefore, the dimension is $10 - 6 = 4$.

$\square$

(L-10) Question 19(a) Since the right hand side vector $\boldsymbol{b}$ belongs to $\mathbb{R}^3$, then A has three rows. In addition, $\boldsymbol{x}$ also belongs to $\mathbb{R}^3$, thus A has also three columns.

Besides, there are two special solutions; therefore $\operatorname{rg}\left(\mathsf{A}\right) = 3 - 2 = 1$. It follows that there is only one pivot row, hence $\dim \mathcal{C}\left(\mathsf{A}^\mathsf{T}\right) = 1$.

$\square$

(L-10) Question 19(b) From the particular solution, it follows that twice the first column equals the right hand side vector $\begin{pmatrix} 2 \\ 4 \\ 2 \end{pmatrix}$, hence, the first column of A is $\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$.

Because the rank is one, the other columns are multiples of the first one. From the first special solution we know that the second column must be the opposite of the first one, or $\begin{pmatrix} -1 \\ -2 \\ -1 \end{pmatrix}$. Finally, from the second special solution it follows that the last column is the zero vector. Consequently,

$$\mathsf{A} = \begin{bmatrix} 1 & -1 & 0 \\ 2 & -2 & 0 \\ 1 & -1 & 0 \end{bmatrix}.$$

$\square$

(L-10) Question 19(c) For any vector $\boldsymbol{b}$ in the column space of A; in other words, the system is solvable for any for any multiple of the first column.

$\square$

(L-10) Question 20(a) **False.** Example: $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$

$\square$

(L-10) Question 20(b) **False.** Example: $\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$

$$\mathcal{C}\left(\mathsf{A}\right) = \left\{ \boldsymbol{x} \in \mathbb{R}^2 \;\middle|\; \exists\, c \in \mathbb{R} \text{ such that } \boldsymbol{x} = c \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}.$$

$$\mathcal{C}\left(\mathsf{A}^\mathsf{T}\right) = \left\{ \boldsymbol{x} \in \mathbb{R}^2 \;\middle|\; \exists\, c \in \mathbb{R} \text{ such that } \boldsymbol{x} = c \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}.$$

$\square$

(L-10) Question 20(c) **True.** The dimension of both spaces is the number of pivots. It is called the rank of the matrix.

$\square$

(L-10) Question 20(d) **False**. The columns can be linearly dependent.

$\square$

(L-10) Question 21(a) False.

$\square$

(L-10) Question 21(b) True.

$\square$

(L-10) Question 21(c) Then, the former statements are false in general.

$\square$

(L-Opt-1) Question 1(a) The subspace $\mathbb{R}^{3\times 3}$ of all matrices 3 by 3, since any matrix can be express as a sum of a symmetric and a triangular matrices.

$\square$

(L-Opt-1) Question 1(b) The intesection of both: the set of all diagonal matrices

$\square$

(L-Opt-1) Question 2(a) **False.** Consider for example

$$\begin{bmatrix}1&0&0\\0&1&0\\0&0&0\end{bmatrix}+\begin{bmatrix}0&0&0\\0&0&0\\0&0&1\end{bmatrix}=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$$

The matrix on the right hand side is invertible, but the two on the left hand side are not.

□

(L-Opt-1) Question 2(b) **True.** For $\mathbf{A}\boldsymbol{x}=\boldsymbol{b}$ to have no solution we must have a row of 0's in the reduced row echelon form. Hence, the number of pivots will be less than the number of rows, and so the matrix $\mathbf{A}$ does not have full rank.

□

(L-Opt-1) Question 2(c) **False.** Suppose $\mathbf{AB}$ is invertible, and consider $\mathbf{C}=(\mathbf{AB})^{-1}\,\mathbf{A}$. Then $\mathbf{CB}=(\mathbf{AB})^{-1}\mathbf{AB}=\mathbf{I}$, so $\mathbf{C}$ is an inverse for $\mathbf{B}$.

Otra demostración alternativa: existe una matriz de rango completo $\mathbf{E}$ tal que $\mathbf{BE}=\mathbf{L}$, donde $\mathbf{L}$ es su forma escalonada. Como $\mathbf{B}$ es singular, $\mathbf{L}$ tiene una columna de ceros, entonces $(\mathbf{AB})\mathbf{E}=\mathbf{AL}=\mathbf{M}$ tiene necesariamente una columna de ceros como $\mathbf{L}$ y por tanto $\mathbf{AB}$ es singular.

□

(L-Opt-1) Question 2(d) **False.** Consider the permutation matrix

$$\mathbf{P}=\begin{bmatrix}0&0&1\\1&0&0\\0&1&0\end{bmatrix}$$

Then

$$\mathbf{P}^2=\begin{bmatrix}0&0&1\\1&0&0\\0&1&0\end{bmatrix}\begin{bmatrix}0&0&1\\1&0&0\\0&1&0\end{bmatrix}=\begin{bmatrix}0&1&0\\0&0&1\\1&0&0\end{bmatrix}\neq\mathbf{I}.$$

□

(L-Opt-1) Question 3(a) Es un sub-espacio de $\mathbb{R}^7$ de dimensión:
cero si los tres son vectores nulos;
uno si $\boldsymbol{v}$ y $\boldsymbol{w}$ son múltiplos de $\boldsymbol{u}$;
dos si como máximo se pueden elegir dos vectores linealmente independientes;
o tres si los tres vectores son linealmente independientes.

□

(L-Opt-1) Question 3(b) También es únicamente por el vector nulo $\mathbf{0}$.

□

(L-Opt-1) Question 3(c) No. Por ejemplo las matrices 5 por 5 identidad ($\mathbf{I}$) y la matriz opuesta a la identidad ($-\mathbf{I}$) son de rango completo, y por tanto son invertibles, pero su suma es la matriz nula, que no es invertible. Por tanto, dicho subconjunto no es cerrado para la suma, es decir, no es un subespacio vectorial.

$$\mathbf{I}+(-\mathbf{I})=\mathbf{0}\quad\text{que no es invertible.}$$

□

(L-Opt-1) Question 3(d) Falso. Ejemplos

$$\begin{bmatrix}0&1\\0&0\end{bmatrix};\quad\begin{bmatrix}0&0\\1&0\end{bmatrix};\quad\begin{bmatrix}1&1\\-1&-1\end{bmatrix};\quad\begin{bmatrix}-1&-1\\1&1\end{bmatrix}$$

□

(L-Opt-1) Question 3(e) El espacio columna $\mathcal{C}\left(\mathbf{A}\right)$ y el espacio nulo por la izquierda $\mathcal{N}\left(\mathbf{A}^\intercal\right)$.

□

(L-Opt-1) Question 3(f) El espacio fila $\mathcal{C}\left(\mathbf{A}^\intercal\right)$ y el espacio nulo $\mathcal{N}\left(\mathbf{A}\right)$.

□

(L-Opt-1) Question 3(g) Si está en el espacio nulo, implica que sumar a la primera columna dos veces la segunda y tres veces la tercera nos da un vector de ceros.

$$\mathbf{A}\begin{pmatrix}1\\2\\3\end{pmatrix}=\mathbf{0}$$

Que es algo incompatible si el vector $\begin{pmatrix}1, & 2, & 3\end{pmatrix}$ es una fila,

$$\begin{bmatrix}1 & 2 & 3\\ - & - & -\\ - & - & -\end{bmatrix}\begin{pmatrix}1\\2\\3\end{pmatrix}\neq\mathbf{0}!$$

□

(L-Opt-1) Question 4(a) Es espacio vectorial ya que

$$\begin{bmatrix}a & b\\0 & b\end{bmatrix}+\begin{bmatrix}c & d\\0 & d\end{bmatrix}=\begin{bmatrix}a+c & b+d\\0 & b+d\end{bmatrix}\text{ pertecene al conjunto; y}$$

$$c\begin{bmatrix}a & b\\0 & b\end{bmatrix}=\begin{bmatrix}ca & cd\\0 & cd\end{bmatrix}\text{ también pertecene al conjunto.}$$

□

(L-Opt-1) Question 4(b) No es espacio vectorial. Sean $g(\cdot)$, $h(\cdot)$ dos funciones de dicho conjunto, entonces la suma evaluada en cero es $g(0)+h(0)=4$, y por tanto no pertenece al conjunto.

□

(L-Opt-1) Question 5. {Todos los monomios (vectores) de la forma: $a_n x^n$ para $n=0,1,2...$}

□

(L-Opt-1) Question 6(a) **Tres.** Puesto que las esquinas superior-derecha e inferior-izquierda deben ser iguales, sólo tres variables pueden variar.

□

(L-Opt-1) Question 6(b) **Tres.** Puesto que podemos re-escribir las matrices como

$$\mathbf{A}=\begin{bmatrix}a & b\\c & -a\end{bmatrix},$$

está claro que el espacio es de dimensión tres como en el caso anterior.

□

(L-Opt-1) Question 6(c) **Dos.** Sólo x e y pueden variar, de hecho, dicho conjunto lo podemos expresar como:

$$\left\{\boldsymbol{v}\in\mathbb{R}^4\;\middle|\;\exists\boldsymbol{p}\in\mathbb{R}^2,\ \boldsymbol{v}=\begin{bmatrix}1 & 0\\0 & 1\\1 & -3\\-1 & 2\end{bmatrix}\boldsymbol{p}\right\}$$

□