Mathematics II

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You can find the last version of these course materials at

https://github.com/mbujosab/MatematicasII/tree/main/Eng

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Part IV
Ortogonality

LECTURE 11: Orthogonal vectors and subspaces

Lecture 11

(**Lecture** 11) S-1 Highlights of Lesson 11

Highlights of *Lesson 11*

- Orthogonal vectors and subspaces
- Nullspace ⊥ row space

$$\mathcal{N}(\mathbf{A}) \perp \mathcal{C}(\mathbf{A}^\top)$$

- left nullspace ⊥ column space

$$\mathcal{N}(\mathbf{A}^\top) \perp \mathcal{C}(\mathbf{A})$$

- From parametric to Cartesian (or implicit) equations

F1

(**Lecture** 11) S-2 Some definitions

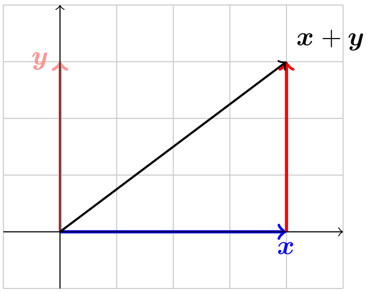
- Dot product

$$\boldsymbol{a} \cdot \boldsymbol{b} = \sum_{i=1}^{n} a_i b_i$$

- Length of a vector $\|\boldsymbol{a}\| = \sqrt{\boldsymbol{a} \cdot \boldsymbol{a}}$ $\qquad \boldsymbol{a} \cdot \boldsymbol{a} = \|\boldsymbol{a}\|^2.$
- Unit vector: $\|\boldsymbol{a}\| = 1$ $\qquad \frac{1}{\|\boldsymbol{x}\|} \cdot \boldsymbol{x}$
- Orthogonal (perpendicular) vectors: $\boldsymbol{x} \cdot \boldsymbol{y} = 0.$

F2

(**Lecture** 11) S-**3** Orthogonal vectors



$$\boldsymbol{x}\cdot\boldsymbol{y} = 0 \iff \boldsymbol{x}\perp\boldsymbol{y}$$

Pythagoras Thm.: $$\boldsymbol{x}\cdot\boldsymbol{y} = 0 \iff \|\boldsymbol{x}\|^2+\|\boldsymbol{y}\|^2 = \|\boldsymbol{x}+\boldsymbol{y}\|^2$$

$$\boldsymbol{x}\cdot\boldsymbol{x}+\boldsymbol{y}\cdot\boldsymbol{y} = (\boldsymbol{x}+\boldsymbol{y})\cdot(\boldsymbol{x}+\boldsymbol{y}).$$

F3

(**Lecture** 11) S-**4** Squared length of a vector

$$\|\boldsymbol{v}\|^2 = \boldsymbol{v}\cdot\boldsymbol{v}$$

$$\boldsymbol{x} = \begin{pmatrix}1\\2\\3\end{pmatrix} \to \|\boldsymbol{x}\|^2 = \quad ; \qquad \boldsymbol{y} = \begin{pmatrix}2\\-1\\0\end{pmatrix} \to \|\boldsymbol{y}\|^2 = \quad ;$$

Are these vectors orthogonal?

$$\boldsymbol{x}+\boldsymbol{y} = \begin{pmatrix} \\ \\ \end{pmatrix}; \qquad \|\boldsymbol{x}+\boldsymbol{y}\|^2 = \quad ;$$

(Pythagoras) $\boldsymbol{x}\cdot\boldsymbol{x}+\boldsymbol{y}\cdot\boldsymbol{y} = (\boldsymbol{x}+\boldsymbol{y})\cdot(\boldsymbol{x}+\boldsymbol{y})$ $\iff$ (Orthogonality) $\boldsymbol{x}\cdot\boldsymbol{y} = 0.$

F4

(**Lecture** 11) S-**5** Orthogonal subspaces

When subspace $\mathcal{S}$ is orthogonal to subspace $\mathcal{T}$:

Every vector in $\mathcal{S}$ is orthogonal to every vector in $\mathcal{T}$

Are the plane of the *blackboard* and the floor orthogonal?

F5

(**Lecture** 11) S-6 Nullspace orthogonal to row space

- $\mathcal{N}(\mathbf{A}) \perp$ rows of $\mathbf{A}$

$$\mathbf{A}\boldsymbol{x} = \mathbf{0} \quad \Longrightarrow \quad \begin{pmatrix} ({}_{1|}\mathbf{A}) \cdot \boldsymbol{x} \\ \vdots \\ ({}_{m|}\mathbf{A}) \cdot \boldsymbol{x} \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$$

- $\mathcal{N}(\mathbf{A}) \perp \boldsymbol{d}\mathbf{A}, \quad \forall \boldsymbol{d} \in \mathbb{R}^m$ (any linear combination of the rows)

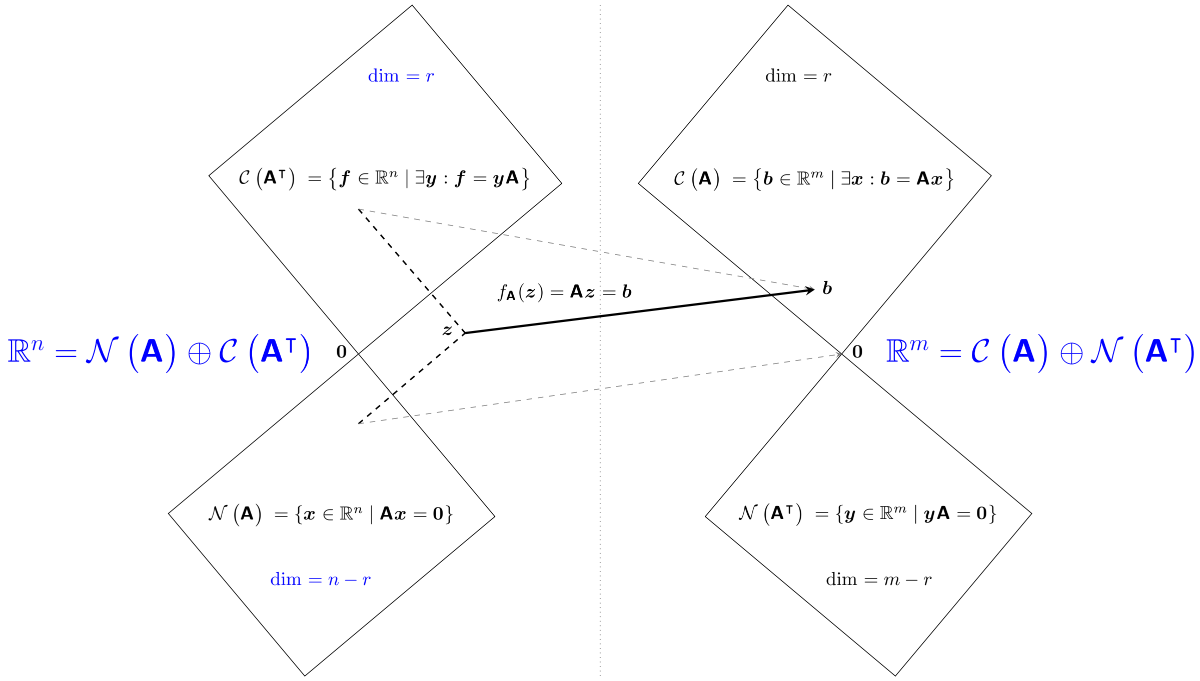
$$\boldsymbol{x} \in \mathcal{N}(\mathbf{A}) \quad \Rightarrow \quad \boldsymbol{d}\mathbf{A}\boldsymbol{x} = \boldsymbol{d} \cdot \mathbf{0} = 0.$$

nullspace $\perp$ row space	$\mathcal{N}(\mathbf{A}) \perp \mathcal{C}(\mathbf{A}^\intercal)$

Also: $\boldsymbol{x}\mathbf{A} = \mathbf{0} \quad \Rightarrow \quad \mathcal{N}(\mathbf{A}^\intercal) \perp \mathcal{C}(\mathbf{A})$

F6

(**Lecture** 11) S-7 The big picture: direct sum of orthogonal complements



$$\mathcal{C}(\mathbf{A}^\intercal) \perp \mathcal{N}(\mathbf{A}) \qquad \mathcal{C}(\mathbf{A}) \perp \mathcal{N}(\mathbf{A}^\intercal)$$

$$\boldsymbol{f} \cdot \boldsymbol{x} = \boldsymbol{y}\mathbf{A}\boldsymbol{x} = \boldsymbol{y} \cdot \mathbf{0} \qquad \boldsymbol{y} \cdot \boldsymbol{b} = \boldsymbol{y}\mathbf{A}\boldsymbol{x} = \mathbf{0} \cdot \boldsymbol{x}$$

F7

(**Lecture** 11) S-8 **Revisiting the Gaussian elimination**

It's an algorithm to find a basis for the orthogonal complement

Give me some vectors (I write them as rows of $\mathbf{M}$) and ...

$$\left[\frac{\mathbf{M}}{\mathbf{I}}\right] = \left[\begin{array}{cccc} 1 & -3 & 0 & -1 \\ 0 & -1 & 1 & 1 \\ 1 & -4 & 1 & 0 \\ \hline 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow[{[(1)\mathbf{2}+\mathbf{4}]}]{\substack{\tau \\ [(3)\mathbf{1}+\mathbf{2}] \\ [(1)\mathbf{1}+\mathbf{4}] \\ [(1)\mathbf{2}+\mathbf{3}]}} \left[\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ \hline 1 & 3 & 3 & 4 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right] = \left[\frac{\mathbf{L}}{\mathbf{E}}\right] = \left[\begin{array}{cc} \mathbf{C} & \mathbf{0} \\ \hline \mathbf{D} & \mathbf{N} \end{array}\right]$$

Basis for the span of the given (row) vectors: $\mathcal{V}$

Basis for orthogonal complement: $\mathcal{V}^{\perp}$ $\mathbf{MN} = \mathbf{0}$

If you had given me $\mathbf{N}_{|1}$ and $\mathbf{N}_{|2}$, after Gaussian elimination would have obtained a basis for...

F8

(**Lecture** 11) S-9 **Cartesian (implicit) and parametric equations of lines and planes**

Cartesian (implicit) equations $\{\boldsymbol{x} \in \mathbb{R}^n \mid \mathbf{A}\boldsymbol{x} = \boldsymbol{b}\}$:

For example

$$\left\{ \boldsymbol{x} \in \mathbb{R}^3 \;\middle|\; \begin{bmatrix} 1 & -1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \boldsymbol{x} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\} = \text{sol. set of } \begin{cases} x_1 - x_2 + x_3 = 1 \\ x_3 = 1 \end{cases}$$

Parametric equations:

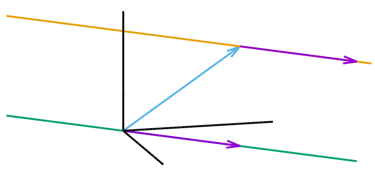
for the above set

$$\left\{ \boldsymbol{x} \in \mathbb{R}^3 \;\middle|\; \exists \boldsymbol{p} \in \mathbb{R}^1 : \boldsymbol{x} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \boldsymbol{p} \right\}$$

In this case *dimension* 1 A line (there is only one parameter a)

line line

F9

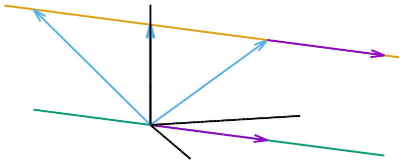


or

$$\left\{ \boldsymbol{x} \in \mathbb{R}^3 \;\middle|\; \exists \boldsymbol{p} \in \mathbb{R}^1 : \boldsymbol{x} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \boldsymbol{p} \right\}$$

or

$$\left\{ \boldsymbol{x} \in \mathbb{R}^3 \;\middle|\; \exists \boldsymbol{p} \in \mathbb{R}^1 : \boldsymbol{x} = \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} + \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \boldsymbol{p} \right\}$$



(**Lecture** 11) S-**10** Cartesian (implicit) and parametric equations of lines and planes

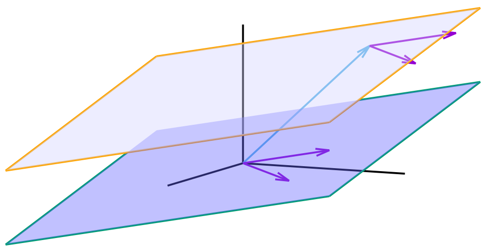
Cartesian (implicit) equations $\{\boldsymbol{x} \in \mathbb{R}^n \mid \mathbf{A}\boldsymbol{x} = \boldsymbol{b}\}$**:**
For example

$$\left\{\boldsymbol{x} \in \mathbb{R}^3 \;\middle|\; \begin{bmatrix}1 & -1 & 1\end{bmatrix} \boldsymbol{x} = (1,) \right\} = \text{sol. set of } \left\{x_1 - x_2 + x_3 = 1\right.$$

Parametric equations:
for the above set

$$\left\{\boldsymbol{x} \in \mathbb{R}^3 \;\middle|\; \exists \boldsymbol{p} \in \mathbb{R}^2 : \boldsymbol{x} = \begin{pmatrix}0\\0\\1\end{pmatrix} + \begin{bmatrix}1 & -1\\1 & 0\\0 & 1\end{bmatrix}\boldsymbol{p}\right\}$$

In this case *dimension* 2 A plane (two parameters a and b)
plane plane F11

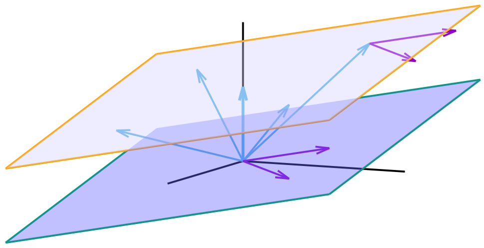


or

$$\left\{\boldsymbol{x} \in \mathbb{R}^3 \;\middle|\; \exists \boldsymbol{p} \in \mathbb{R}^2 : \boldsymbol{x} = \begin{pmatrix}1\\1\\1\end{pmatrix} + \begin{bmatrix}1 & -1\\1 & 0\\0 & 1\end{bmatrix}\boldsymbol{p}\right\}$$

but also

$$\left\{\boldsymbol{x} \in \mathbb{R}^3 \;\middle|\; \exists \boldsymbol{p} \in \mathbb{R}^2 : \boldsymbol{x} = \begin{pmatrix}-1\\-1\\1\end{pmatrix} + \begin{bmatrix}1 & -1\\1 & 0\\0 & 1\end{bmatrix}\boldsymbol{p}\right\}$$



(**Lecture** 11) S-**11** From parametric to Cartesian equations

$$\boxed{\mathcal{C}\left(\mathbf{A}^{\intercal}\right) \perp \mathcal{N}\left(\mathbf{A}\right)}$$

Consider

$$H = \left\{\boldsymbol{x} \in \mathbb{R}^n \;\middle|\; \exists \boldsymbol{p} \in \mathbb{R}^k : \boldsymbol{x} = \boldsymbol{s} + [\boldsymbol{n}_1; \;\dots\; \boldsymbol{n}_k;]\boldsymbol{p}\right\}.$$

If we find $\mathbf{A}$ *such that* $\mathbf{A}\boldsymbol{n}_i = \mathbf{0}$ then if $\boldsymbol{x} \in H$

$$\mathbf{A}\boldsymbol{x} \;=\; \mathbf{A}\boldsymbol{s} \;+\; \underbrace{\mathbf{A}\,[\boldsymbol{n}_1; \;\dots\; \boldsymbol{n}_k;]}_{\mathbf{0}}\,\boldsymbol{p} \quad\Rightarrow\quad \mathbf{A}\boldsymbol{x} \;=\; \boldsymbol{b}, \quad \text{where } \boldsymbol{b} = \mathbf{A}\boldsymbol{s}.$$

Therefore

$$H = \{\boldsymbol{x} \in \mathbb{R}^n \mid \mathbf{A}\boldsymbol{x} = \boldsymbol{b}\}.$$

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(**Lecture** 11) S-**12** From the set of solution to a linear system

Find the implicit equations of the plane P parallel to the spam of $(1,2,0,-2)$ and $(0,0,1,3)$, that goes through $\boldsymbol{s} = (1,3,1,1)$.

$$P = \left\{ \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} \;\middle|\; \exists a, b \in \mathbb{R} : \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 1 \\ 1 \end{pmatrix} + a \begin{pmatrix} 1 \\ 2 \\ 0 \\ -2 \end{pmatrix} + b \begin{pmatrix} 0 \\ 0 \\ 1 \\ 3 \end{pmatrix} \right\}$$

$$= \left\{ \boldsymbol{x} \in \mathbb{R}^4 \;\middle|\; \exists \boldsymbol{p} \in \mathbb{R}^2 : \boldsymbol{x} = \begin{pmatrix} 1 \\ 3 \\ 1 \\ 1 \end{pmatrix} + \begin{bmatrix} 1 & 0 \\ 2 & 0 \\ 0 & 1 \\ -2 & 3 \end{bmatrix} \boldsymbol{p} \right\}$$

We need vectors perpendicular to $(1,2,0,-2)$ and $(0,0,1,3)$

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(Lecture 11) S-13 From the set of solution to a linear system

$\boldsymbol{x} = (x, y, z, w,);\qquad \boldsymbol{s} = (1, 3, 1, 1,).$

$$\left[\begin{array}{cccc} 1 & 2 & 0 & -2 \\ 0 & 0 & 1 & 3 \\ \hline x & y & z & w \\ \hline 1 & 3 & 1 & 1 \end{array}\right] \xrightarrow[{[(2)\mathbf{1}+\mathbf{4}]}]{\substack{\tau \\ {[(-2)\mathbf{1}+\mathbf{2}]}}} \left[\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 3 \\ \hline x & y-2x & z & w+2x \\ \hline 1 & 1 & 1 & 3 \end{array}\right] \xrightarrow[{[(-3)\mathbf{3}+\mathbf{4}]}]{\tau} \left[\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \hline x & y-2x & z & w+2x-3z \\ \hline 1 & 1 & 1 & 0 \end{array}\right]$$

So $\mathbf{A} = \begin{bmatrix} -2 & 1 & 0 & 0 \\ 2 & 0 & -3 & 1 \end{bmatrix}$; and then $\mathbf{A}\boldsymbol{x} = \begin{pmatrix} -2x+y \\ 2x+w-3z \end{pmatrix}$ and $\boldsymbol{b} = \mathbf{A}\boldsymbol{s} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Hence $\begin{cases} -2x + y & = 1 \\ 2x \quad -3z + w & = 0 \end{cases}$

$$P = \left\{ \boldsymbol{x} \in \mathbb{R}^4 \;\middle|\; \begin{bmatrix} -2 & 1 & 0 & 0 \\ 2 & 0 & -3 & 1 \end{bmatrix} \boldsymbol{x} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}.$$

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The lecture ends here

Questions of the Lecture 11

(L-11) QUESTION 1. Describe the set of vectors in $\mathbb{R}^3$ orthogonal to this one $\begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix}$

(Hefferon, 2008, exercise 2.15 from section II.2.)

(L-11) QUESTION 2.

(a) Find a parametric representation for the line passing through the points $\boldsymbol{x}_P = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ y $\boldsymbol{x}_Q = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$.

(b) Find a implicit representation for the same line.

(L-11) QUESTION 3.

(a) Find a parametric representation for the line passing through the points $\boldsymbol{x}_P = (1, \quad -3, \quad 1,)$ and $\boldsymbol{x}_Q = (-2, \quad 4, \quad 5,)$.

(b) Find a implicit representation (Cartesian equations) for the same line.

(Lang, 1986, Example 1 in Section 1.5)

(L-11) QUESTION 4. Is there any vector perpendicular to itself?

(Hefferon, 2008, exercise 2.17 from section II.2.)

(L-11) QUESTION 5.

(a) Parametric equation of a line parallel to $2x - 3y = 5$ that goes through $(1, 1)$.

(b) Find a implicit representation for the line.

(L-11) QUESTION 6. Find the length of each vector

(a) $\begin{pmatrix} 1 \\ 3 \end{pmatrix}$. (b) $\begin{pmatrix} -1 \\ 2 \end{pmatrix}$. (c) $\begin{pmatrix} 4 \\ 1 \\ 1 \end{pmatrix}$.

(d) $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$. (e) $\begin{pmatrix} 1 \\ -1 \\ 1 \\ 0 \end{pmatrix}$.

(Hefferon, 2008, exercise 2.11 from section II.2.)

(L-11) QUESTION 7. Find a unit vector with the same direction as $\boldsymbol{v} = (2, -1, 0, 4, -2)$.

(L-11) Question 8. Find k so that these two vectors are perpendicular.

$$\begin{pmatrix} k, & 1, \end{pmatrix}, \qquad \begin{pmatrix} 4, & 3, \end{pmatrix}.$$

(Hefferon, 2008, exercise 2.14 from section II.2.)

(L-11) Question 9. Construc a matrix with the required property or say why that is impossible:

(a) Column space contains $\begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ -3 \\ 5 \end{pmatrix}$, nullspace contains $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

(b) Row space contains $\begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ -3 \\ 5 \end{pmatrix}$, and nullspace contains $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

(c) $\mathbf{A}\boldsymbol{x} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ has a solution and $\mathbf{A}^{\mathsf{T}} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

(d) Every row is orthogonal to every column ($\mathbf{A}$ is not the zero matrix)

(e) Columns add up to a column of zeros, rows add up to a row of 1's.

(Strang, 2003, exercise 3 from section 4.1.)

(L-11) Question 10. If $\mathbf{AB} = \mathbf{0}$, the columns of $\mathbf{B}$ are in the ________ of $\mathbf{A}$. The rows of $\mathbf{A}$ are in the __________ of $\mathbf{B}$. Why can't $\mathbf{A}$ and $\mathbf{B}$ be 3 by 3 matrices of rank 2?
(Strang, 2003, exercise 4 from section 4.1.)

(L-11) Question 11. Suppose that $\boldsymbol{u} \cdot \boldsymbol{v} = \boldsymbol{u} \cdot \boldsymbol{w}$ and $\boldsymbol{u} \neq \mathbf{0}$. Must $\boldsymbol{v} = \boldsymbol{w}$?
(Hefferon, 2008, exercise 2.20 from section II.2.)

(L-11) Question 12.

(a) If $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$ has a solution and $\mathbf{A}^{\mathsf{T}}\boldsymbol{y} = \mathbf{0}$, then $\boldsymbol{y}$ is perpendicular to ____.

(b) If $\mathbf{A}^{\mathsf{T}}\boldsymbol{y} = \boldsymbol{c}$ has a solution and $\mathbf{A}\boldsymbol{x} = \mathbf{0}$, then $\boldsymbol{x}$ is perpendicular to ____.

(Strang, 2003, exercise 5 from section 4.1.)

(L-11) Question 13. Demuestre, in $\mathbb{R}^n$, that if $\boldsymbol{u}$ and $\boldsymbol{v}$ are perpendicular then $||\boldsymbol{u} + \boldsymbol{v}||^2 = ||\boldsymbol{u}||^2 + ||\boldsymbol{v}||^2$.
(Hefferon, 2008, exercise 2.33 from section II.2.)

(L-11) Question 14.

(a) Find parametric equations of the plane that goes through the point (0,1,1) and parallel to the vectors (0,1,2) and (1,1,0)

(b) Write the implicit equation of the same plane.

(L-11) Question 15.

(a) Find a parametric equation of the plane through the point $\begin{pmatrix} 2, & 1, & 3, \end{pmatrix}$ with normal vector $\begin{pmatrix} 3, & 1, & 1, \end{pmatrix}$.

(b) Write the implicit equation of the same plane.

(L-11) Question 16. Find a 1 by 3 matrix whose nullspace consists of all vectors in $\mathbb{R}^3$ such that $x_1 + 2x_2 + 4x_3 = 0$. Find a 3 by 3 matrix with that same nullspace.
(Strang, 2006, exercise 9 from section 2.4.)

(L-11) Question 17. Consider the system $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$, where

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 0 & 1 & 1 \\ 0 & 0 & 2 & 3 & 1 \\ 0 & 0 & 1 & 4 & 2 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}, \qquad \boldsymbol{b} = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 2 \end{pmatrix}.$$

(a) (1^{pts}) Find the solution to the system.

(b) (0.5^{pts}) Explain why the solution set is a line in $\mathbb{R}^5$. Find a direction vector (a vector parallel to the line) and any point on that line.

(c) (1^{pts}) Find the set of vectors perpendicular to the solution set. Prove that set is a four dimensional subspace. Find a basis for that subspace.

(L-11) Question 18. Consider $\underset{4\times 2}{\mathbf{A}}$ with exactly two special solutions for $\boldsymbol{x}\mathbf{A} = \mathbf{0}$:

$\boldsymbol{s}_1 = \begin{pmatrix}3, & 1, & 0, & 0,\end{pmatrix}$, and $\boldsymbol{s}_2 = \begin{pmatrix}6, & 0, & 2, & 1,\end{pmatrix}$.

(a) Find the reduced row echelon form $\mathbf{R}$ of $\mathbf{A}$.

(b) What is the row space of $\mathbf{A}$?

(c) What is the complete solution to $\boldsymbol{x}\mathbf{R} = \begin{pmatrix}3, & 6,\end{pmatrix}$?

(d) Find a combination of rows 2, 3, 4 that equals $\mathbf{0}$. (Not OK to use $0({}_{2|}\mathbf{A}) + 0({}_{3|}\mathbf{A}) + 0({}_{4|}\mathbf{A})$. The problem is to show that these rows are dependent.)

basado en MIT Course 18.06 Quiz 1, March 4, 2013

End of Questions of the Lecture 11

LECTURE 12: Projections onto subspaces

Lecture 12

(**Lecture** 12) S-1 Highlights of Lesson 12

Highlights of *Lesson 12*

- Projections
- Projection matrices

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(**Lecture** 12) S-2 Direct sum of subspaces

$\mathbb{R}^n$ is a *direct sum* of $\mathcal{A}$ and $\mathcal{B}$ $\quad (\mathbb{R}^n = \mathcal{A} \oplus \mathcal{B})$

if every $\boldsymbol{x} \in \mathbb{R}^n$ has a **unique** representation $\boldsymbol{x} = \boldsymbol{a} + \boldsymbol{b}$,

with $\boldsymbol{a} \in \mathcal{A}$ and $\boldsymbol{b} \in \mathcal{B}$.

Example 1. $\boxed{\mathbb{R}^n = \mathcal{C}\left(\mathbf{A}^\intercal\right) \oplus \mathcal{N}\left(\mathbf{A}\right)}$

$$\left[\frac{\mathbf{A}}{\mathbf{I}}\right] = \left[\begin{array}{ccc} 1 & 2 & 5 \\ 2 & 4 & 10 \\ \hline 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc} 1 & 0 & 0 \\ 2 & 0 & 0 \\ \hline 1 & -2 & -5 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right] \Rightarrow \text{Basis of } \mathbb{R}^3; \left[\begin{pmatrix} 1 \\ 2 \\ 5 \end{pmatrix}; \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}; \begin{pmatrix} -5 \\ 0 \\ 1 \end{pmatrix}\right]$$

$$\forall \boldsymbol{x} \in \mathbb{R}^3, \ \exists c_1, c_2, c_3 \ \Big| \ \boldsymbol{x} = c_1 \begin{pmatrix} 1 \\ 2 \\ 5 \end{pmatrix} + c_2 \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} + c_3 \begin{pmatrix} -5 \\ 0 \\ 1 \end{pmatrix} = \boldsymbol{a} + \boldsymbol{b}$$

where $\boldsymbol{a} \in \mathcal{C}\left(\mathbf{A}^\intercal\right)$ and $\boldsymbol{b} \in \mathcal{N}\left(\mathbf{A}\right)$.

Also $\boxed{\mathbb{R}^m = \mathcal{C}\left(\mathbf{A}\right) \oplus \mathcal{N}\left(\mathbf{A}^\intercal\right)}$

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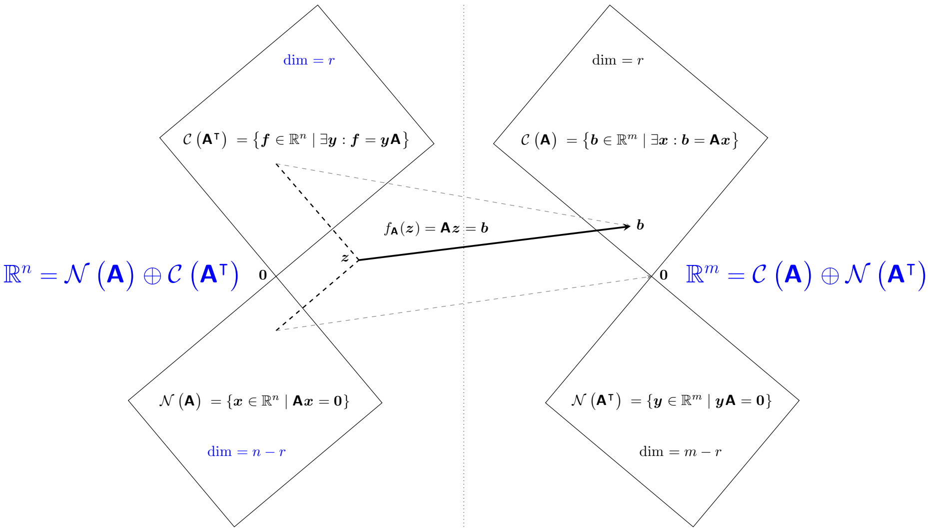
(**Lecture** 12) S-3 The big picture: direct sum of orthogonal complements



$$\mathcal{C}(\mathbf{A}^\intercal) \perp \mathcal{N}(\mathbf{A}) \qquad\qquad \mathcal{C}(\mathbf{A}) \perp \mathcal{N}(\mathbf{A}^\intercal)$$

$$\boldsymbol{f} \cdot \boldsymbol{x} = \boldsymbol{y}\mathbf{A}\boldsymbol{x} = \boldsymbol{y} \cdot \mathbf{0} \qquad\qquad \boldsymbol{y} \cdot \boldsymbol{b} = \boldsymbol{y}\mathbf{A}\boldsymbol{x} = \mathbf{0} \cdot \boldsymbol{x}$$

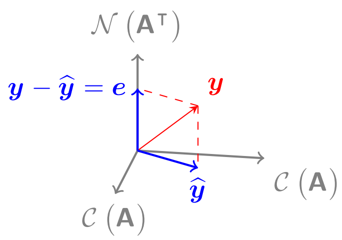
F18

(**Lecture** 12) S-4 Orthogonal Projection onto $\mathcal{C}(\mathbf{A})$

Consider $\underset{m\times n}{\mathbf{A}}$; since $\mathbb{R}^m = \mathcal{C}(\mathbf{A}) \oplus \mathcal{N}(\mathbf{A}^\intercal)$, for any $\boldsymbol{y} \in \mathbb{R}^m$

$$\boldsymbol{y} = \widehat{\boldsymbol{y}} + \boldsymbol{e}; \qquad (\boldsymbol{e} = \boldsymbol{y} - \widehat{\boldsymbol{y}})$$

where $\boxed{\widehat{\boldsymbol{y}} \in \mathcal{C}(\mathbf{A}) \text{ and } \boldsymbol{e} \perp \widehat{\boldsymbol{y}}}$, so $\boldsymbol{e} \in \mathcal{N}(\mathbf{A}^\intercal)$.



How to compute $\widehat{\boldsymbol{y}} \in \mathcal{C}(\mathbf{A})$?

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(**Lecture** 12) S-5 Normal equations

Consider $\underset{m\times n}{\mathbf{A}}$. We want to find the descoposition $\boxed{\boldsymbol{y}=\widehat{\boldsymbol{y}}+\boldsymbol{e}}$ where

$$\widehat{\boldsymbol{y}}\in\mathcal{C}\left(\mathbf{A}\right)\qquad\text{and}\qquad(\widehat{\boldsymbol{y}}-\boldsymbol{y})\in\mathcal{N}\left(\mathbf{A}^\intercal\right)$$

Then

$$\mathbf{A}\widehat{\boldsymbol{x}}=\widehat{\boldsymbol{y}}\qquad\Leftrightarrow\qquad(\mathbf{A}\widehat{\boldsymbol{x}}-\boldsymbol{y})\in\mathcal{N}\left(\mathbf{A}^\intercal\right)$$

Therefore

$$\mathbf{A}\widehat{\boldsymbol{x}}=\widehat{\boldsymbol{y}}\quad\Leftrightarrow\quad\mathbf{A}^\intercal\left(\mathbf{A}\widehat{\boldsymbol{x}}-\boldsymbol{y}\right)=\mathbf{0}\quad\Leftrightarrow\quad\boxed{(\mathbf{A}^\intercal\mathbf{A})\widehat{\boldsymbol{x}}=\mathbf{A}^\intercal\boldsymbol{y}}$$

Equivalent systems! $\Rightarrow\ \mathcal{N}\left(\mathbf{A}\right)=\mathcal{N}\left(\mathbf{A}^\intercal\mathbf{A}\right)\ \Rightarrow\ \operatorname{rg}(\mathbf{A})=\operatorname{rg}(\mathbf{A}^\intercal\mathbf{A})$

unique solution $\widehat{\boldsymbol{x}}$ if and only if $\mathbf{A}$ is full column rank

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(**Lecture** 12) S-6 The solution to the normal equations (full column rank)

$$\boxed{\mathbf{A}^\intercal\mathbf{A}\widehat{\boldsymbol{x}}=\mathbf{A}^\intercal\boldsymbol{y}}\qquad(\mathbf{A}\text{ is full column rank})$$

The solution $\widehat{\boldsymbol{x}}=\ (\mathbf{A}^\intercal\mathbf{A})^{-1}\mathbf{A}^\intercal\boldsymbol{y}$

The projection $\widehat{\boldsymbol{y}}=\mathbf{A}\widehat{\boldsymbol{x}}=\mathbf{A}(\mathbf{A}^\intercal\mathbf{A})^{-1}\mathbf{A}^\intercal\boldsymbol{y}$

The projection matrix $\mathbf{P}=\mathbf{A}(\mathbf{A}^\intercal\mathbf{A})^{-1}\mathbf{A}^\intercal$

$$\widehat{\boldsymbol{y}}=\mathbf{P}\boldsymbol{y}$$

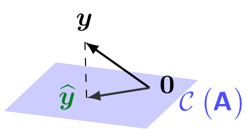
$\mathbf{P}$: Symetric and idempotent.

F21

(**Lecture** 12) S-7 Projection matrix

$$\boxed{\mathbf{P}=\mathbf{A}\left(\mathbf{A}^\intercal\mathbf{A}\right)^{-1}\mathbf{A}^\intercal}$$

Projection $\mathbf{P}\boldsymbol{y}$ *is the point* $\widehat{\boldsymbol{y}}$ *of* $\mathcal{C}\left(\mathbf{A}\right)$ *closest to* $\boldsymbol{y}$

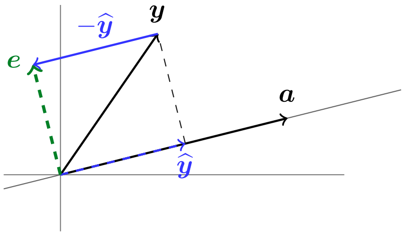


Extreme cases:

- If $\boldsymbol{y}\in\mathcal{C}\left(\mathbf{A}\right)$ then $\mathbf{P}\boldsymbol{y}=$
- If $\boldsymbol{y}\perp\mathcal{C}\left(\mathbf{A}\right)$ then $\mathbf{P}\boldsymbol{y}=$

F22

(**Lecture** 12) S-8 **Projection onto a line**



line$= \mathcal{C}(\mathbf{A})$; $\mathbf{A} = [a]$; $\boldsymbol{a} \neq \mathbf{0}$

I'd like to find the point $\widehat{\boldsymbol{y}}$ on that line closest to $\boldsymbol{y}$

$\widehat{\boldsymbol{y}} \in \mathcal{C}([\boldsymbol{a}]) \quad \perp \quad \boldsymbol{e} = (\boldsymbol{y} - \widehat{\boldsymbol{y}}) \in \mathcal{N}([\boldsymbol{a}]^{\mathsf{T}})$.

$\widehat{\boldsymbol{y}}$ is some multiple of $\boldsymbol{a}$: $\widehat{\boldsymbol{y}} = [\boldsymbol{a}](\widehat{x},)$

How: $[\boldsymbol{a}]^{\mathsf{T}}[\boldsymbol{a}]\widehat{\boldsymbol{x}} = [\boldsymbol{a}]^{\mathsf{T}}\boldsymbol{y}$

The solution $\widehat{\boldsymbol{x}} = ([\boldsymbol{a}]^{\mathsf{T}}[\boldsymbol{a}])^{-1}[\boldsymbol{a}]^{\mathsf{T}}\boldsymbol{y}$

The projection $\widehat{\boldsymbol{y}} = [\boldsymbol{a}]\widehat{\boldsymbol{x}} = [\boldsymbol{a}]([\boldsymbol{a}]^{\mathsf{T}}[\boldsymbol{a}])^{-1}[\boldsymbol{a}]^{\mathsf{T}}\boldsymbol{y}$

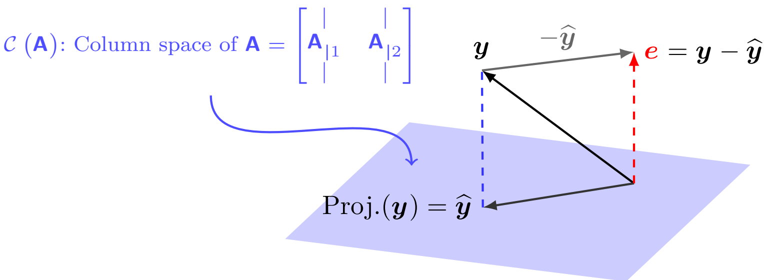
The projection matrix $\mathbf{P} = [\boldsymbol{a}]([\boldsymbol{a}]^{\mathsf{T}}[\boldsymbol{a}])^{-1}[\boldsymbol{a}]^{\mathsf{T}}$

F23

(**Lecture** 12) S-9 **Projection onto a plane**

Why project?
So we will solve

$$\mathbf{A}\boldsymbol{x} = \left(\text{Proj. of } \boldsymbol{y} \text{ onto } \mathcal{C}(\mathbf{A})\right).$$

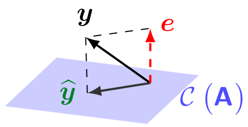


$(\boldsymbol{y} - \widehat{\boldsymbol{y}}) = \boldsymbol{e} \perp \mathcal{C}(\mathbf{A})$ ... that's the crucial fact.

F24

(**Lecture** 12) S-**10** Normal equations

What's the projection of $\boldsymbol{y}$ onto the column space of $\mathbf{A} = \begin{bmatrix} | & | \\ \mathbf{A}_{|1} & \mathbf{A}_{|2} \\ | & | \end{bmatrix}$?



$$\widehat{\boldsymbol{y}} = (\mathbf{A}_{|1})\widehat{x_1} + (\mathbf{A}_{|2})\widehat{x_2} \;=\; \mathbf{A}\widehat{\boldsymbol{x}}$$

"Find the right combination of the columns so $\boldsymbol{e} \perp \mathcal{C}(\mathbf{A})$*"*

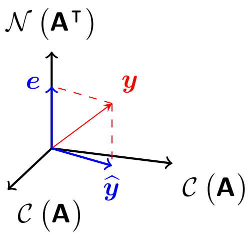
$$\boldsymbol{e} \perp \mathcal{C}(\mathbf{A}) \quad \Rightarrow \quad \boldsymbol{e} \in$$

$$\mathbf{A}^{\mathsf{T}}\boldsymbol{e} = \mathbf{A}^{\mathsf{T}}(\boldsymbol{y} - \widehat{\boldsymbol{y}}) \quad = \quad \mathbf{A}^{\mathsf{T}}(\boldsymbol{y} - \mathbf{A}\widehat{\boldsymbol{x}}) = \mathbf{0} \quad \Leftrightarrow \quad \boxed{(\mathbf{A}^{\mathsf{T}}\mathbf{A})\widehat{\boldsymbol{x}} = \mathbf{A}^{\mathsf{T}}\boldsymbol{y}}$$

F25

(**Lecture** 12) S-**11** Two projections

$\boldsymbol{y}$ has a component $\widehat{\boldsymbol{y}}$ in $\mathcal{C}(\mathbf{A})$, and another component $\boldsymbol{e}$ in $\mathcal{C}(\mathbf{A})^{\perp}$.



$$\widehat{\boldsymbol{y}} + \boldsymbol{e} = \boldsymbol{y}$$

$$\widehat{\boldsymbol{y}} = \mathbf{P}\boldsymbol{y} \qquad \text{projection onto } \mathcal{C}(\mathbf{A})$$

$$\boldsymbol{e} = (\mathbf{I} - \mathbf{P})\boldsymbol{y} \qquad \text{projection onto } \mathcal{C}(\mathbf{A})^{\perp}$$

F26

The lecture ends here

Questions of the Lecture 12

(L-12) Question 1. Project the first vector orthogonally into the line spanned by the second vector. Check that $\boldsymbol{e}$ is perpendicular to $\boldsymbol{a}$. Find the projection matrix $\mathbf{P} = [\boldsymbol{a}]\left([\boldsymbol{a}]^{\mathsf{T}}[\boldsymbol{a}]\right)^{-1}[\boldsymbol{a}]^{\mathsf{T}}$ onto the line through each vector $\boldsymbol{a}$. Verify in each case that $\mathbf{P}^2 = \mathbf{P}$. Multiply $\mathbf{P}\boldsymbol{b}$ in each case to compute the projection $\widehat{\boldsymbol{b}}$.

(a) $\boldsymbol{b} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$; $\boldsymbol{a} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$.

(b) $\boldsymbol{b} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$; $\boldsymbol{a} = \begin{pmatrix} 3 \\ 0 \end{pmatrix}$.

(c) $\boldsymbol{b} = \begin{pmatrix} 1 \\ 1 \\ 4 \end{pmatrix}$; $\boldsymbol{a} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$.

(d) $\boldsymbol{b} = \begin{pmatrix} 1 \\ 1 \\ 4 \end{pmatrix}$; $\boldsymbol{a} = \begin{pmatrix} 3 \\ 3 \\ 12 \end{pmatrix}$.

(Hefferon, 2008, exercise 1.6 from section VI.1.)

(L-12) Question 2. Project the vector orthogonally into the line.

(a) $\begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix}$, The line : $\left\{ \boldsymbol{v} \in \mathbb{R}^3 \;\middle|\; \exists \boldsymbol{p} \in \mathbb{R}^1,\ \boldsymbol{v} = \begin{bmatrix} -3 \\ 1 \\ -3 \end{bmatrix} \boldsymbol{p} \right\}$.

(b) $\begin{pmatrix} -1 \\ -1 \end{pmatrix}$, the line $y = 3x$.

(L-12) Question 3. Although pictures guided our development, we are not restricted to spaces that we can draw. In $\mathbb{R}^4$ project this vector into this line.

$$\begin{pmatrix} 1 \\ 2 \\ 1 \\ 3 \end{pmatrix} ; \quad \left\{ \boldsymbol{v} \in \mathbb{R}^4 \;\middle|\; \exists \boldsymbol{p} \in \mathbb{R}^1,\ \boldsymbol{v} = \begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix} \boldsymbol{p} \right\}.$$

(L-12) Question 4.

(a) Project the vector $\boldsymbol{b} = (1, \ 1,)$ onto the lines through $\boldsymbol{a}_1 = (1, \ 0,)$ and $\boldsymbol{a}_2 = (1, \ 2,)$. Add the projections: $\widehat{\boldsymbol{b}_1} + \widehat{\boldsymbol{b}_2}$. The projections do not add to $\boldsymbol{b}$ because $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$ are not orthogonal.

(b) The projection of $\boldsymbol{b}$ onto the plane of $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$ will equal $\boldsymbol{b}$. Find $\mathbf{P} = \mathbf{A}(\mathbf{A}^\mathsf{T}\mathbf{A})^{-1}\mathbf{A}^\mathsf{T}$ for $\mathbf{A} = [\boldsymbol{a}_1; \boldsymbol{a}_2;]$.

(Strang, 2003, exercise 8–9 from section 4.2.)

(L-12) Question 5.

(a) If $\mathbf{P}^2 = \mathbf{P}$ show that $(\mathbf{I} - \mathbf{P})^2 = \mathbf{I} - \mathbf{P}$. When $\mathbf{P}$ projects onto the column space of $\mathbf{A}$, $(\mathbf{I} - \mathbf{P})$ projects onto the ________.

(b) If $\mathbf{P}^\mathsf{T} = \mathbf{P}$ show that $(\mathbf{I} - \mathbf{P})^\mathsf{T} = \mathbf{I} - \mathbf{P}$.

(Strang, 2003, exercise 17 from section 4.2.)

(L-12) Question 6.

(a) Compute the projection matrices $\mathbf{P} = [\boldsymbol{a}]\left([\boldsymbol{a}]^\mathsf{T}[\boldsymbol{a}]\right)^{-1}[\boldsymbol{a}]^\mathsf{T}$ onto the lines through $\boldsymbol{a}_1 = (-1, \ 2, \ 2,)$ and $\boldsymbol{a}_2 = (2, \ 2, \ -1,)$. Show that $\boldsymbol{a}_1 \perp \boldsymbol{a}_2$. Multiply those projection matrices and explain why their product $\mathbf{P}_1\mathbf{P}_2$ is what it is.

(b) Project $\boldsymbol{b} = (1, \ 0, \ 0,)$ onto the lines through $\boldsymbol{a}_1$, and $\boldsymbol{a}_2$ and also onto $\boldsymbol{a}_3 = (2, \ -1, \ 2,)$. Add up the three projections $\widehat{\boldsymbol{b}_1} + \widehat{\boldsymbol{b}_2} + \widehat{\boldsymbol{b}_3}$.

(c) Find the projection matrix $\mathbf{P}_3$ onto $\mathcal{L}([\boldsymbol{a}_3;]) = \mathcal{L}([(2, \ -1, \ 2,);])$. Verify that $\mathbf{P}_1 + \mathbf{P}_2 + \mathbf{P}_3 = \mathbf{I}$. The basis $\boldsymbol{a}_1$, $\boldsymbol{a}_2$, $\boldsymbol{a}_3$ is orthogonal!

(Strang, 2003, exercise 5–7 from section 4.2.)

(L-12) Question 7. Project $\boldsymbol{b}$ onto the column space of $\mathbf{A}$ by solving $\mathbf{A}^\mathsf{T}\mathbf{A}\widehat{\boldsymbol{x}} = \mathbf{A}^\mathsf{T}\boldsymbol{b}$ and then computing $\widehat{\boldsymbol{b}} = \mathbf{A}\widehat{\boldsymbol{x}}$. Find $\boldsymbol{e} = \boldsymbol{b} - \widehat{\boldsymbol{b}}$.

(a) $\mathbf{A}_1 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $\boldsymbol{b}_1 = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}$

(b) $\mathbf{A}_2 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}$ and $\boldsymbol{b}_2 = \begin{pmatrix} 4 \\ 4 \\ 6 \end{pmatrix}$

(c) Compute the projection matrices $\mathbf{P}_1$ and $\mathbf{P}_2$ onto the column spaces. Verify that $\mathbf{P}_1\boldsymbol{b}_1$ gives the first projection $\widehat{\boldsymbol{b}_1}$. Also verify $(\mathbf{P}_2)^2 = \mathbf{P}_2$.

(Strang, 2003, exercise 11–12 from section 4.2.)

End of Questions of the Lecture 12

References

Hefferon, J. (2008). *Linear Algebra*. Jim Hefferon, Colchester, Vermont USA. This text is Free. URL `ftp://joshua.smcvt.edu/pub/hefferon/book/book.pdf`

Lang, S. (1986). *Introduction to Linear Algebra*. Springer-Verlag, second ed.

Strang, G. (2003). *Introduction to Linear Algebra*. Wellesley-Cambridge Press, Wellesley, Massachusetts. USA, third ed. ISBN 0-9614088-9-8.

Strang, G. (2006). *Linear algebra and its applications*. Thomson Learning, Inc., fourth ed. ISBN 0-03-010567-6.

Solutions

(L-11) Question 1. Since $\mathcal{C}\left(\mathbf{A}^{\intercal}\right) \perp \mathcal{N}\left(\mathbf{A}\right)$, we only need to find the orthogonal complement of the spam of $\begin{bmatrix}1 & 3 & -1\end{bmatrix}$.

$$\left[\frac{\mathbf{A}}{\mathbf{I}}\right] = \left[\begin{array}{ccc}1 & 3 & -1\\ \hline 1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{array}\right] \xrightarrow[{[(1)\mathbf{1}+\mathbf{3}]}]{\overset{\tau}{[(-3)\mathbf{1}+\mathbf{2}]}} \left[\begin{array}{ccc}1 & 0 & 0\\ \hline 1 & -3 & 1\\ 0 & 1 & 0\\ 0 & 0 & 1\end{array}\right] = \left[\frac{\mathbf{L}}{\mathbf{E}}\right]$$

Therefore, the set of vectors in $\mathbb{R}^3$ orthogonal to $\begin{pmatrix}1, & 3, & -1\end{pmatrix}$ is

$$\left\{ \boldsymbol{v} \in \mathbb{R}^3 \;\middle|\; \exists \boldsymbol{p} \in \mathbb{R}^2 \text{ tal que } \boldsymbol{v} = \begin{bmatrix}-3 & 1\\ 1 & 0\\ 0 & 1\end{bmatrix} \boldsymbol{p} \right\}.$$

With NAcAL there are several ways to obtain such a subspace. There are two ways to invoke `Subspace`: if the argument is a system (`Sistema`) of `Vector`s of $\mathbb{R}^n$, it returns the subspace spanned by that system.

```
a = Vector([-3,1,0])
b = Vector([1,0,1])
SubEspacio(Sistema([a,b]))
```

If the argument is a `Matrix`, it returns it's null space.

```
v = Vector([1,3,-1])
A = ~Matrix([v])     # trasponemos para obtener la matriz fila
SubEspacio(A)
```

But since we are asked for the orthogonal complement of the subspace generated by the vector, we can simply write (since in this context means the orthogonal complement):

```
~SubEspacio(Sistema([v]))
```

The representation by means of parametric or Cartesian equations is not unique, in fact, we obtain different parametric equations for the systems $[\boldsymbol{a};\boldsymbol{b};]$ (seen above) and $[\boldsymbol{b};\boldsymbol{a};]$.

```
SubEspacio(Sistema([b,a]))
```

It is therefore useful to be able to check whether two subspaces are equal

```
~SubEspacio(Sistema([v])) == SubEspacio(Sistema([b,a]))
```

□

(L-11) Question 2(a) We first have to find a vector parallel to the line. We let

$$\boldsymbol{v} = \boldsymbol{x}_P - \boldsymbol{x}_Q = \begin{pmatrix}1\\2\end{pmatrix} + \begin{pmatrix}3\\1\end{pmatrix} = \begin{pmatrix}-2\\1\end{pmatrix}.$$

A parametric representation of the line is therefore

$$\left\{ \boldsymbol{v} \in \mathbb{R}^2 \;\middle|\; \exists \boldsymbol{p} \in \mathbb{R}^1,\ \boldsymbol{v} = \begin{pmatrix}1\\2\end{pmatrix} + \begin{bmatrix}-2\\1\end{bmatrix} \boldsymbol{p} \right\}.$$

With NAcAL, points, lines, planes, etc. (i.e. planar regions in R[n]) are created with `EAfin`. The required arguments to `EAfin` are a `Subspace` and a `Vector`. If instead of a `Subspace` a `System` of `Vectors` of R[n] or a `Matrix` is given, NAcAL shall use those arguments to generate the necessary subspace (the subspace generated by the system in in the first case, or the null space of the matrix in the second).

Thus, in this case we obtain the equations of the required line with:

```
p = Vector([1,2])
q = Vector([3,1])
S = SubEspacio(Sistema([p-q]))
R = EAfin(S,p)
Math( R.EcParametricas() )  # Por ahora solo quiero visualizar las Ec. Paramétricas de R
```

□

(L-11) Question 2(b) We need to multiply $\boldsymbol{x} = \boldsymbol{x}_P + a\boldsymbol{v}$ by a vector perpendicular to $\boldsymbol{v}$. We will do it by elimination:

$$\left[\begin{array}{cc} -2 & 1 \\ \hline x & y \\ \hline 1 & 2 \end{array}\right] \xrightarrow[{[(1)\mathbf{1}+\mathbf{2}]}]{\substack{\boldsymbol{\tau} \\ [(2)\mathbf{2}]}} \left[\begin{array}{cc} -2 & 0 \\ \hline x & x+2y \\ \hline 1 & 5 \end{array}\right] \Rightarrow \text{ the solution set of } \left\{ x + 2y = 5 \right. ;$$

y therefore the line is:

$$\left\{ \boldsymbol{v} \in \mathbb{R}^2 \;\middle|\; \begin{bmatrix} 1 & 2 \end{bmatrix} \boldsymbol{v} = (5,) \right\}.$$

Let's reproduce the pencil and paper calculation with NAcAL.

```
x,y = sympy.symbols('x y')
N = ~Matrix([p-q])
M = N.apila(~Matrix([Vector([x,y])]),1).apila(~Matrix([p]),1)
Math( rprElim(M, Elim(N).pasos) )
```

Therefore the straight line is the set of vectors that solve the following system of linear equations:

```
A = Matrix([[1,2]])
b = Vector([5])
SEL(A,b).eafin
```

(note that NAcAL stores as an attribute (of type `EAfin`) the set of solutions to a system of equations)

□

(L-11) Question 3(a) We first have to find a vector in the direction of the line. We let

$$\boldsymbol{v} = \boldsymbol{x}_P - \boldsymbol{x}_Q = \begin{pmatrix} 1 \\ -3 \\ 1 \end{pmatrix} - \begin{pmatrix} -2 \\ 4 \\ 5 \end{pmatrix} = (3, \quad -7, \quad -4,).$$

A parametric representation of the line is therefore

$$\left\{ \boldsymbol{v} \in \mathbb{R}^3 \;\middle|\; \exists \boldsymbol{p} \in \mathbb{R}^1,\ \boldsymbol{v} = (1, \quad -3, \quad 1,) + \begin{bmatrix} 3 \\ -7 \\ -4 \end{bmatrix} \boldsymbol{p} \right\}.$$

□

(L-11) Question 3(b)

$$\left[\begin{array}{ccc} 3 & -7 & -4 \\ \hline x & y & z \\ \hline 1 & -3 & 1 \end{array}\right] \xrightarrow[{[(4)\mathbf{1}+\mathbf{3}]}]{\substack{\boldsymbol{\tau} \\ [(3)\mathbf{2}] \\ [(7)\mathbf{1}+\mathbf{2}] \\ [(3)\mathbf{3}]}} \left[\begin{array}{ccc} 3 & 0 & 0 \\ \hline x & 7x+3y & 4x+3z \\ \hline 1 & -2 & 7 \end{array}\right] \Rightarrow \begin{cases} 7x + 3y & = -2 \\ 4x \qquad + 3z & = 7 \end{cases};$$

Por tanto las ecuaciones cartesianas de la recta son:

$$\left\{ \boldsymbol{v} \in \mathbb{R}^3 \;\middle|\; \begin{bmatrix} 7 & 3 & 0 \\ 4 & 0 & 3 \end{bmatrix} \boldsymbol{v} = \begin{pmatrix} -2 \\ 7 \end{pmatrix} \right\}.$$

This system has two equations. If we take them separately, they correspond to two planes in $\mathbb{R}^3$.

```
p1=SEL(Matrix([[7,3,0]]),Vector([-2])).eafin
p1
```

and

```
p1=SEL(Matrix([[7,3,0]]),Vector([-2])).eafin
p1
```

(we know that they are two planes, because the parametric equations have two parameters, and the coefficient matrices of the Cartesian equations have two free columns) The line of the exercise corresponds to the intersection of both planes, that is, to the points that belong to both planes:

```
p1 & p2
```

□

(L-11) Question 4. $\boldsymbol{a}\cdot\boldsymbol{a} = \sum_{i=1}^{n} a_i^2 = 0 \iff \boldsymbol{a} = \boldsymbol{0}$. Therefore, the answer is yes: the zero vector $\boldsymbol{0}$.

□

(L-11) Question 5(a) Since it is parallel to the line $2x - 3y = 5$, we need to find a vector $\boldsymbol{v}$ in the nullspace of the coeficient matrix of the system, for example:

$$\left[\begin{array}{cc} 2 & -3 \\ \hline 1 & 0 \\ 0 & 1 \end{array}\right] \xrightarrow[{[(2)\mathbf{2}]}]{\substack{\tau \\ {[(6)\mathbf{1}]}}} \left[\begin{array}{cc} 6 & -6 \\ \hline 3 & 0 \\ 0 & 2 \end{array}\right] \xrightarrow{\substack{\tau \\ {[(1)\mathbf{2}+\mathbf{1}]}}} \left[\begin{array}{cc} 6 & 0 \\ \hline 3 & 3 \\ 0 & 2 \end{array}\right]$$

therefore $$\left\{ \boldsymbol{x} \in \mathbb{R}^2 \;\middle|\; \exists a \in \mathbb{R} \text{ tal que } \boldsymbol{x} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} + a \begin{pmatrix} 3 \\ 2 \end{pmatrix} \right\}$$

□

(L-11) Question 5(b) We only need to substitute (x, y) by $(1, 1)$ to obtain the right hand side "vector" $\boldsymbol{b}$.

$$2x - 3y = 2\cdot 1 - 3\cdot 1 = -1 \quad\Rightarrow\quad 2x - 3y = -1.$$

Hence

$$\left\{ \begin{pmatrix} x \\ y \end{pmatrix} \;\middle|\; \begin{bmatrix} 2 & -3 \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 \end{pmatrix} \right\}$$

□

(L-11) Question 6(a) $\sqrt{3^2 + 1^2} = \sqrt{10}$

□

(L-11) Question 6(b) $\sqrt{5}$

□

(L-11) Question 6(c) $\sqrt{18}$

□

(L-11) Question 6(d) 0

□

(L-11) Question 6(e) $\sqrt{3}$

□

(L-11) Question 7. $\|\boldsymbol{v}\|^2 = \boldsymbol{v}\cdot\boldsymbol{v} = 4 + 1 + 0 + 16 + 4 = 25$ so we take $\boldsymbol{u} = \frac{1}{\|\boldsymbol{v}\|}\cdot\boldsymbol{v} = (\frac{2}{5}, \frac{-1}{5}, 0, \frac{4}{5}, \frac{-2}{5})$.

□

(L-11) Question 8. Its dot product must be zero, therefore $(k)(4) + (1)(3) = 0$ therefore $k = -3/4$.

□

(L-11) Question 9(a) $\begin{bmatrix} 1 & 2 & a \\ 2 & -3 & b \\ -3 & 5 & c \end{bmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix} + \begin{pmatrix} 2 \\ -3 \\ 5 \end{pmatrix} + \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} -3 \\ 1 \\ -2 \end{pmatrix}$;

So, $\mathsf{A} = \begin{bmatrix} 1 & 2 & -3 \\ 2 & -3 & 1 \\ -3 & 5 & -2 \end{bmatrix}$.

□

(L-11) Question 9(b) Impossible, $\begin{pmatrix}2\\-3\\5\end{pmatrix}$ not orthogonal to $\begin{pmatrix}1\\1\\1\end{pmatrix}$

□

(L-11) Question 9(c) $\begin{pmatrix}1\\1\\1\end{pmatrix}$ in $\mathcal{C}\left(\mathbf{A}\right)$, and $\begin{pmatrix}1\\0\\0\end{pmatrix}$ in $\mathcal{N}\left(\mathbf{A}^{\mathsf{T}}\right)$. It is impossible: these vectors are not perpendicular.

□

(L-11) Question 9(d) This asks for $\mathbf{A}\cdot\mathbf{A}=\mathbf{0}$. Take, for example $\mathbf{A}=\begin{bmatrix}1&-1\\1&-1\end{bmatrix}$, or $\mathbf{A}=\begin{bmatrix}0&0\\1&0\end{bmatrix}$ for example.

□

(L-11) Question 9(e) $\left(1,\quad 1,\quad 1,\right)$ will be in the nullspace, $\mathbf{A}\begin{pmatrix}1\\1\\1\end{pmatrix}=\mathbf{0}$; and row space, $\left(1,\quad 1,\quad 1,\right)\mathbf{A}=\left(1,\quad 1,\quad 1,\right)$, …no such matrix.

□

(L-11) Question 10. If $\mathbf{AB}=\mathbf{0}$, the columns of $\mathbf{B}$ are in the *nullspace* of $\mathbf{A}$. The rows of $\mathbf{A}$ are in the *left nullspace* of $\mathbf{B}$.

If rank $=2$, all four subspaces would have dimension 2 which is impossible for 3 by 3 matrix.

□

(L-11) Question 11. No. These give an example.

$$\boldsymbol{u}=\begin{pmatrix}1\\0\end{pmatrix}\quad \boldsymbol{v}=\begin{pmatrix}1\\0\end{pmatrix}\quad \boldsymbol{w}=\begin{pmatrix}1\\1\end{pmatrix},$$

$\boldsymbol{u}\cdot\boldsymbol{v}=1=\boldsymbol{u}\cdot\boldsymbol{w}$, but $\boldsymbol{v}\neq\boldsymbol{w}$.

□

(L-11) Question 12(a) On the one hand $\mathbf{A}\boldsymbol{x}=\boldsymbol{b}\Rightarrow\boldsymbol{b}\in\mathcal{C}\left(\mathbf{A}\right)$ on the other hand $\mathbf{A}^{\mathsf{T}}\boldsymbol{y}=\mathbf{0}\Rightarrow\boldsymbol{y}\perp\in\mathcal{C}\left(\mathbf{A}\right)$.

If $\mathbf{A}\boldsymbol{x}=\boldsymbol{b}$ has a solution and $\mathbf{A}^{\mathsf{T}}\boldsymbol{y}=\mathbf{0}$, then $\boldsymbol{y}$ is perpendicular to $\boldsymbol{b}$.

$$\boldsymbol{y}\cdot\boldsymbol{b}=\boldsymbol{y}\mathbf{A}\boldsymbol{b}=\mathbf{0}\cdot\boldsymbol{b}=0.$$

□

(L-11) Question 12(b) If $\mathbf{A}^{\mathsf{T}}\boldsymbol{y}=\boldsymbol{c}$ then $\boldsymbol{y}\mathbf{A}=\boldsymbol{c}$, also $\mathbf{A}\boldsymbol{x}=\mathbf{0}$; then $\boldsymbol{x}$ is perpendicular to $\boldsymbol{c}$.

$\boldsymbol{c}$ is in the row space, and therefore it is orthogonal to $\boldsymbol{x}$, that is a vector in the nullspace. In other words:

$$\boldsymbol{c}\cdot\boldsymbol{x}=\boldsymbol{y}\mathbf{A}\boldsymbol{x}=\boldsymbol{y}\cdot\mathbf{0}=0.$$

□

(L-11) Question 13. If $\boldsymbol{u}$ and $\boldsymbol{v}$ are perpendicular then

$$\|(\boldsymbol{u}+\boldsymbol{v})\|^2=(\boldsymbol{u}+\boldsymbol{v})\cdot(\boldsymbol{u}+\boldsymbol{v})=\boldsymbol{u}\cdot\boldsymbol{u}+2\left(\boldsymbol{u}\cdot\boldsymbol{v}\right)+\boldsymbol{v}\cdot\boldsymbol{u}=\boldsymbol{u}\cdot\boldsymbol{u}+\boldsymbol{v}\cdot\boldsymbol{v}=\|\boldsymbol{u}\|^2+\|\boldsymbol{v}\|^2$$

(the third equality holds because $\boldsymbol{u}\cdot\boldsymbol{v}=0$).

□

(L-11) Question 14(a) $\left\{\boldsymbol{v}\in\mathbb{R}^3\;\middle|\;\exists\boldsymbol{p}\in\mathbb{R}^2,\ \boldsymbol{v}=\left(0,\quad 1,\quad 1,\right)+\begin{bmatrix}0&1\\1&1\\2&0\end{bmatrix}\boldsymbol{p}\right\}.$

□

(L-11) Question 14(b)

$$\left[\begin{array}{ccc}0&1&2\\1&1&0\\\hline x&y&z\\\hline 0&1&1\end{array}\right]\xrightarrow{[(-1)\mathbf{1}+\mathbf{2}]^{\tau}}\left[\begin{array}{ccc}0&1&2\\1&0&0\\\hline x&-x+y&z\\\hline 0&1&1\end{array}\right]\xrightarrow{[(-2)\mathbf{2}+\mathbf{3}]^{\tau}}\left[\begin{array}{ccc}0&1&0\\1&0&0\\\hline x&-x+y&2x-2y+z\\\hline 0&1&-1\end{array}\right]$$

Therefore: $\left\{\boldsymbol{v}\in\mathbb{R}^3 \;\middle|\; \begin{bmatrix}2 & -2 & 1\end{bmatrix}\boldsymbol{v}=\begin{pmatrix}-1,\end{pmatrix}\right\}.$

```
p = Vector([0,1,1])
v = Vector([0,1,2])
w = Vector([1,1,0])
S = SubEspacio(Sistema([v,w]))
EAfin(S,p)
```

□

(L-11) Question 15(a) Since the plane is in the 3 dimensinal space $\mathbb{R}^3$, in this case we need to find two vectors orthohonal to $\begin{pmatrix}3, & 1, & 1,\end{pmatrix}$. For example, $\begin{pmatrix}-1, & 3, & 0,\end{pmatrix}$ and $\begin{pmatrix}0, & -1, & 1,\end{pmatrix}$. therefore,

$$\left\{\boldsymbol{x}\in\mathbb{R}^3 \;\middle|\; \exists\boldsymbol{p}\in\mathbb{R}^2 \text{ tal que } \boldsymbol{x}=\begin{pmatrix}2\\1\\3\end{pmatrix}+\begin{bmatrix}1 & 0\\-3 & -1\\0 & 1\end{bmatrix}\boldsymbol{p}\right\}.$$

□

(L-11) Question 15(b) In this case we already know a vector orthogonal to the parametric part, hence:

$$\begin{bmatrix}3 & 1 & 1\end{bmatrix}\boldsymbol{x}=\begin{bmatrix}3 & 1 & 1\end{bmatrix}\boldsymbol{s}; \quad\Rightarrow\quad \begin{bmatrix}3 & 1 & 1\end{bmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=\begin{bmatrix}3 & 1 & 1\end{bmatrix}\begin{pmatrix}2\\1\\3\end{pmatrix}=(10,);$$

$$\Longrightarrow \quad \left\{\boldsymbol{x}\in\mathbb{R}^2 \;\middle|\; \begin{bmatrix}3 & 1 & 1\end{bmatrix}\boldsymbol{x}=\begin{pmatrix}10,\end{pmatrix}\right\}.$$

```
p = Vector([2,1,3])
v = Vector([3,1,1])
S =  SubEspacio(~Matrix([v])) # esta es una alternativa
#S = ~SubEspacio(Sistema([v])) # esta es otra alternativa
EAfin(S,p)
```

□

(L-11) Question 16. We can take as row of $\mathbf{A}$, a linear combination of a basis of the left null space of $\begin{bmatrix}1\\2\\4\end{bmatrix}$. Hence,

$$\left[\begin{array}{ccc}1 & 2 & 4\\\hline 1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{array}\right]\xrightarrow[{[(-4)\mathbf{1}+\mathbf{3}]}]{\substack{\tau\\{[(-2)\mathbf{1}+\mathbf{2}]}}}\left[\begin{array}{ccc}1 & 0 & 0\\\hline 1 & -2 & -4\\0 & 1 & 0\\0 & 0 & 1\end{array}\right] \quad\Rightarrow\quad \begin{bmatrix}-2 & 1 & 0\\-4 & 0 & 1\end{bmatrix}$$

and then

$$\underset{1\times 3}{\mathbf{A}}=\begin{bmatrix}1 & 1\end{bmatrix}\begin{bmatrix}-2 & 1 & 0\\-4 & 0 & 1\end{bmatrix}=\begin{bmatrix}-6 & 1 & 1\end{bmatrix}$$

but also

$$\mathbf{C}=\begin{bmatrix}1 & 0\\0 & 1\\1 & 1\end{bmatrix}\begin{bmatrix}-2 & 1 & 0\\-4 & 0 & 1\end{bmatrix}=\begin{bmatrix}-2 & 1 & 0\\-4 & 0 & 1\\6 & -1 & -1\end{bmatrix}$$

has the same nullspace...

□

(L-11) Question 17(a) La solución completa es:

$$\boldsymbol{b}=\left\{\boldsymbol{v}\in\mathbb{R}^5 \;\middle|\; \exists\boldsymbol{p}\in\mathbb{R}^1,\; \boldsymbol{v}=\begin{pmatrix}-1\\0\\1\\-2\\4\end{pmatrix}+\begin{bmatrix}-2\\1\\0\\0\\0\end{bmatrix}\boldsymbol{p}\right\}$$

```
A = Matrix([ [1,2,0,1,1], [0,0,2,3,1], [0,0,1,4,2], [0,0,0,1,1] ])
b = Vector( [1,0,1,2] )
SEL (A, b, 1)
```

□

(L-11) Question 17(b) Puesto que la matriz de coeficientes tiene cinco columnas, el sistema tiene cinco incognitas, así pues, los vectores que pertenecen al conjunto de soluciones tienen cinco componentes (un número por columna). Así pues, el conjunto de soluciones es un subconjunto de $\mathbb{R}^5$; Y en este caso, dicho conjunto es una recta, ya que la dimensión de $\mathcal{N}\left(\mathbf{A}\right)$ es uno. Así pues, un vector director es cualquier múltiplo (excepto el vector nulo $\mathbf{0}$) de la solución especial que hemos encontrado: $\boldsymbol{n} = \begin{pmatrix}-2, & 1, & 0, & 0, & 0,\end{pmatrix}$. Y uno de los puntos por donde pasa la recta es la solución particular que obtuvimos al resolver el sistema: $\boldsymbol{s} = \begin{pmatrix}-1, & 0, & 1, & -2, & 4,\end{pmatrix}$.

□

(L-11) Question 17(c)

$$\begin{bmatrix}[\boldsymbol{n}]^\intercal \\ \hline \mathbf{I}\end{bmatrix} = \begin{bmatrix} -2 & 1 & 0 & 0 & 0 \\ \hline 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow[{[(1)\mathbf{1}+\mathbf{2}]}]{\substack{\boldsymbol{\tau} \\ [(2)\mathbf{2}]}} \begin{bmatrix} -2 & 0 & 0 & 0 & 0 \\ \hline 1 & 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix}\mathbf{L} \\ \hline \mathbf{E}\end{bmatrix}$$

Las cuatro últimas columnas de la matriz $\mathbf{E}$ son vectores perpendiculares a $\boldsymbol{n}$; y es evidente que son cuatro, y que son linealmente independientes, así que son una base del subespacio perpendicular a $\boldsymbol{n}$.

□

(L-11) Question 18(a) Any column of $\mathbf{A}$ is orthogonal to the two special solutions given in the problem. That is,

$$\begin{bmatrix} 3 & 1 & 0 & 0 \\ 6 & 0 & 2 & 1 \\ \hline 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{\substack{\boldsymbol{\tau} \\ [(-3)\mathbf{2}+\mathbf{1}] \\ [(-6)\mathbf{4}+\mathbf{1}] \\ [(-2)\mathbf{4}+\mathbf{3}]}} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ \hline 1 & 0 & 0 & 0 \\ -3 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -6 & 0 & -2 & 1 \end{bmatrix} \quad \text{so} \quad \mathbf{R} = \begin{bmatrix} 1 & 0 \\ -3 & 0 \\ 0 & 1 \\ -6 & -2 \end{bmatrix}.$$

□

(L-11) Question 18(b) $\mathbf{R}$ has two pivots, and therefore $\mathbf{A}$ has two pivots and $r(\mathbf{A}) = 2$. Two independent rows in $\mathbb{R}^2$ span $\mathbb{R}^2$, so $\mathcal{C}\left(\mathbf{A}^\intercal\right) = \mathbb{R}^2$.

□

(L-11) Question 18(c) Since rows 1 and 3 are pivot rows, then $\boldsymbol{x}_p = \begin{pmatrix}3, & 0, & 6, & 0,\end{pmatrix}$ is a particular solution, so the complete solution is

$$\left\{ \boldsymbol{v} \in \mathbb{R}^4 \;\middle|\; \exists \boldsymbol{p} \in \mathbb{R}^2,\ \boldsymbol{v} = \begin{pmatrix}3, & 0, & 6, & 0,\end{pmatrix} + \boldsymbol{p}\begin{bmatrix} 3 & 1 & 0 & 0 \\ 6 & 0 & 2 & 1 \end{bmatrix} \right\}$$

since

$$\begin{pmatrix}3, & 0, & 6, & 0,\end{pmatrix}\begin{bmatrix} 1 & 0 \\ -3 & 0 \\ 0 & 1 \\ -6 & -2 \end{bmatrix} = \begin{pmatrix}3, & 6,\end{pmatrix}$$

and

$$\boldsymbol{p}\begin{bmatrix} 3 & 1 & 0 & 0 \\ 6 & 0 & 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -3 & 0 \\ 0 & 1 \\ -6 & -2 \end{bmatrix} = \boldsymbol{p}\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{pmatrix}0, & 0,\end{pmatrix}.$$

□

(L-11) Question 18(d) It is easy to see that

$$-2\begin{pmatrix}-3\\0\end{pmatrix}+2\begin{pmatrix}0\\1\end{pmatrix}+\begin{pmatrix}-6\\-2\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$$

If you dont see that, we can always use gaussian elimination

$$\left[\begin{array}{ccc}-3&0&-6\\0&1&-2\\\hline 1&0&0\\0&1&0\\0&0&1\end{array}\right]\xrightarrow[{[(-2)\mathbf{1}+\mathbf{3}]}]{\boldsymbol{\tau}}\left[\begin{array}{ccc}-3&0&0\\0&1&-2\\\hline 1&0&-2\\0&1&0\\0&0&1\end{array}\right]\xrightarrow[{[(2)\mathbf{2}+\mathbf{3}]}]{\boldsymbol{\tau}}\left[\begin{array}{ccc}-3&0&0\\0&1&0\\\hline 1&0&-2\\0&1&2\\0&0&1\end{array}\right].$$

□

(L-12) Question 1(a)

$$\widehat{\boldsymbol{b}}=[\boldsymbol{a}]\left([\boldsymbol{a}]^{\intercal}[\boldsymbol{a}]\right)^{-1}[\boldsymbol{a}]^{\intercal}\boldsymbol{b}=\begin{bmatrix}3\\2\end{bmatrix}\left(\begin{bmatrix}3&2\end{bmatrix}\begin{bmatrix}3\\2\end{bmatrix}\right)^{-1}\begin{bmatrix}3&2\end{bmatrix}\begin{pmatrix}2\\1\end{pmatrix}=\frac{1}{13}\begin{bmatrix}3\\2\end{bmatrix}\begin{bmatrix}3&2\end{bmatrix}\begin{pmatrix}2\\1\end{pmatrix}=\frac{1}{13}\begin{pmatrix}24\\16\end{pmatrix}$$

$$\boldsymbol{e}=\boldsymbol{b}-\widehat{\boldsymbol{b}}=\begin{pmatrix}2\\1\end{pmatrix}-\frac{1}{13}\begin{pmatrix}24\\16\end{pmatrix}=\frac{1}{13}\begin{pmatrix}2\\-3\end{pmatrix}$$

$$\boldsymbol{a}\cdot\boldsymbol{e}=(3,\quad 2,)\begin{pmatrix}2\\-3\end{pmatrix}\frac{1}{13}=0\frac{1}{13}=0.$$

$$\mathsf{P}=\frac{1}{13}\begin{bmatrix}9&6\\6&4\end{bmatrix};\quad \mathsf{P}^2=\frac{1}{13}\cdot\frac{1}{13}\begin{bmatrix}117&78\\78&52\end{bmatrix}=\frac{1}{13}\begin{bmatrix}9&6\\6&4\end{bmatrix}=\mathsf{P};$$

$$\mathsf{P}\boldsymbol{b}=\frac{1}{13}\begin{bmatrix}9&6\\6&4\end{bmatrix}\begin{pmatrix}2\\1\end{pmatrix}=\frac{1}{13}\begin{pmatrix}24\\16\end{pmatrix}.$$

```
a      = Vector([3,2]);  b = Vector([2,1]);   A = Matrix([a])
P      = A*InvMat((~A)*A)*(~A)
bhat   = P*b;    e = b-bhat
Sistema([bhat,e,P])
```

□

(L-12) Question 1(b)

$$\widehat{\boldsymbol{b}}=[\boldsymbol{a}]\left([\boldsymbol{a}]^{\intercal}[\boldsymbol{a}]\right)^{-1}[\boldsymbol{a}]^{\intercal}\boldsymbol{b}=\begin{bmatrix}3\\0\end{bmatrix}\left(\begin{bmatrix}3&0\end{bmatrix}\begin{bmatrix}3\\0\end{bmatrix}\right)^{-1}\begin{bmatrix}3&0\end{bmatrix}\begin{pmatrix}2\\1\end{pmatrix}=\frac{1}{9}\begin{bmatrix}3\\0\end{bmatrix}\begin{bmatrix}3&0\end{bmatrix}\begin{pmatrix}2\\1\end{pmatrix}=\frac{1}{9}\begin{pmatrix}18\\0\end{pmatrix}=\begin{pmatrix}2\\0\end{pmatrix}.$$

$$\boldsymbol{e}=\boldsymbol{b}-\widehat{\boldsymbol{b}}=\begin{pmatrix}2\\1\end{pmatrix}-\frac{1}{9}\begin{pmatrix}18\\0\end{pmatrix}=\frac{1}{9}\begin{pmatrix}0\\9\end{pmatrix}=\begin{pmatrix}0\\1\end{pmatrix}$$

$$\boldsymbol{a}\cdot\boldsymbol{e}=(3,\quad 0,)\begin{pmatrix}0\\9\end{pmatrix}\frac{1}{9}=0\frac{1}{9}=0.$$

$$\mathsf{P}=\frac{1}{9}\begin{bmatrix}9&0\\0&0\end{bmatrix}=\begin{bmatrix}1&0\\0&0\end{bmatrix};\quad \mathsf{P}^2=\begin{bmatrix}1&0\\0&0\end{bmatrix}\begin{bmatrix}1&0\\0&0\end{bmatrix}=\begin{bmatrix}1&0\\0&0\end{bmatrix}=\mathsf{P};$$

$$\mathsf{P}\boldsymbol{b}=\begin{bmatrix}1&0\\0&0\end{bmatrix}\begin{pmatrix}2\\1\end{pmatrix}=\begin{pmatrix}2\\0\end{pmatrix}.$$

□

(L-12) Question 1(c)

$$\widehat{\boldsymbol{b}} = [\boldsymbol{a}]\left([\boldsymbol{a}]^{\mathsf{T}}[\boldsymbol{a}]\right)^{-1}[\boldsymbol{a}]^{\mathsf{T}}\boldsymbol{b} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}\left(\begin{bmatrix} 1 & 2 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}\right)^{-1}\begin{bmatrix} 1 & 2 & -1 \end{bmatrix}\begin{pmatrix} 1 \\ 1 \\ 4 \end{pmatrix} =$$

$$\frac{1}{6}\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}\begin{bmatrix} 1 & 2 & -1 \end{bmatrix}\begin{pmatrix} 1 \\ 1 \\ 4 \end{pmatrix} = \frac{1}{6}\begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix}$$

$$\boldsymbol{e} = \boldsymbol{b} - \widehat{\boldsymbol{b}} = \begin{pmatrix} 1 \\ 1 \\ 4 \end{pmatrix} - \frac{1}{6}\begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix} = \frac{1}{6}\begin{pmatrix} 7 \\ 8 \\ 23 \end{pmatrix}$$

$$\boldsymbol{a} \cdot \boldsymbol{e} = \left(1, \quad 2, \quad -1,\right)\begin{pmatrix} 7 \\ 8 \\ 23 \end{pmatrix}\frac{1}{6} = 0\frac{1}{6} = 0.$$

$$\mathbf{P} = \frac{1}{6}\begin{bmatrix} 1 & 2 & -1 \\ 2 & 4 & -2 \\ -1 & -2 & 1 \end{bmatrix}; \quad \mathbf{P}^2 = \frac{1}{6} \cdot \frac{1}{6}\begin{bmatrix} 6 & 12 & -6 \\ 12 & 24 & -12 \\ -6 & -12 & 6 \end{bmatrix} = \frac{1}{6}\begin{bmatrix} 1 & 2 & -1 \\ 2 & 4 & -2 \\ -1 & -2 & 1 \end{bmatrix}.$$

$$\mathbf{P}\boldsymbol{b} = \frac{1}{6}\begin{bmatrix} 1 & 2 & -1 \\ 2 & 4 & -2 \\ -1 & -2 & 1 \end{bmatrix}\begin{pmatrix} 1 \\ 1 \\ 4 \end{pmatrix} = \frac{1}{6}\begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix}.$$

□

(L-12) Question 1(d)

$$\widehat{\boldsymbol{b}} = [\boldsymbol{a}]\left([\boldsymbol{a}]^{\mathsf{T}}[\boldsymbol{a}]\right)^{-1}[\boldsymbol{a}]^{\mathsf{T}}\boldsymbol{b} = \begin{bmatrix} 3 \\ 3 \\ 12 \end{bmatrix}\left(\begin{bmatrix} 3 & 3 & 12 \end{bmatrix}\begin{bmatrix} 3 \\ 3 \\ 12 \end{bmatrix}\right)^{-1}\begin{bmatrix} 3 & 3 & 12 \end{bmatrix}\begin{pmatrix} 1 \\ 1 \\ 4 \end{pmatrix} =$$

$$\frac{1}{162}\begin{bmatrix} 3 \\ 3 \\ 12 \end{bmatrix}\begin{bmatrix} 3 & 3 & 12 \end{bmatrix}\begin{pmatrix} 1 \\ 1 \\ 4 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 4 \end{pmatrix}$$

$$\boldsymbol{e} = \boldsymbol{b} - \widehat{\boldsymbol{b}} = \begin{pmatrix} 1 \\ 1 \\ 4 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \\ 4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\boldsymbol{a} \cdot \boldsymbol{e} = \boldsymbol{a} \cdot \mathbf{0} = 0.$$

$$\mathbf{P} = \frac{1}{18}\begin{bmatrix} 1 & 1 & 4 \\ 1 & 1 & 4 \\ 4 & 4 & 16 \end{bmatrix}; \quad \mathbf{P}^2 = \frac{1}{18}\frac{1}{18}\begin{bmatrix} 18 & 18 & 72 \\ 18 & 18 & 72 \\ 72 & 72 & 288 \end{bmatrix} = \mathbf{P};$$

$$\mathbf{P}\boldsymbol{b} = \frac{1}{18}\begin{bmatrix} 1 & 1 & 4 \\ 1 & 1 & 4 \\ 4 & 4 & 16 \end{bmatrix}\begin{pmatrix} 1 \\ 1 \\ 4 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 4 \end{pmatrix}.$$

□

(L-12) Question 2(a)

$$\widehat{\boldsymbol{b}} = [\boldsymbol{a}]\left([\boldsymbol{a}]^{\mathsf{T}}[\boldsymbol{a}]\right)^{-1}[\boldsymbol{a}]^{\mathsf{T}}\boldsymbol{b} = \begin{bmatrix} -3 \\ 1 \\ -3 \end{bmatrix}\left(\begin{bmatrix} -3 & 1 & -3 \end{bmatrix}\begin{bmatrix} -3 \\ 1 \\ -3 \end{bmatrix}\right)^{-1}\begin{bmatrix} -3 & 1 & -3 \end{bmatrix}\begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix} =$$

$$\frac{1}{19}\begin{bmatrix} -3 \\ 1 \\ -3 \end{bmatrix}\begin{bmatrix} -3 & 1 & -3 \end{bmatrix}\begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix} = \frac{1}{19}\begin{bmatrix} 9 & -3 & 9 \\ -3 & 1 & -3 \\ 9 & -3 & 9 \end{bmatrix}\begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix} = \begin{pmatrix} 3 \\ -1 \\ 3 \end{pmatrix}$$

```
b      = Vector([2,-1,4]); a = Vector([-3,1,-3]); A = Matrix([a])
P      = A*InvMat((~A)*A)*(~A)      # Matriz proyección
bhat1  = P*b                        # Alternativa 1
x      = SEL( (~A)*A, (~A)*b ).solP # Solución Ecuaciones Normales
bhat2  = A*x                        # Alternativa 2
Sistema([bhat1,bhat2])
```

□

(L-12) Question 2(b) The line is the set of solutions to $3x - y = 0$:

$$\begin{bmatrix} \mathbf{A} \\ \hline \mathbf{I} \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ \hline 1 & 0 \\ 0 & 1 \end{bmatrix} \xrightarrow[{[(1)\mathbf{1}+\mathbf{2}]}]{\substack{\tau \\ [(3)\mathbf{2}]}} \begin{bmatrix} 3 & 0 \\ \hline 1 & 1 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} \mathbf{L} \\ \hline \mathbf{E} \end{bmatrix};$$

so we should project into the line

$$\text{The line : } \left\{ \boldsymbol{v} \in \mathbb{R}^2 \;\middle|\; \exists \boldsymbol{p} \in \mathbb{R}^1,\ \boldsymbol{v} = \begin{bmatrix} 1 \\ 3 \end{bmatrix} \boldsymbol{p} \right\}$$

$$\widehat{\boldsymbol{b}} = [\boldsymbol{a}]\left([\boldsymbol{a}]^{\intercal}[\boldsymbol{a}]\right)^{-1}[\boldsymbol{a}]^{\intercal}\boldsymbol{b} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}\left(\begin{bmatrix} 1 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 3 \end{bmatrix}\right)^{-1}\begin{bmatrix} 1 & 3 \end{bmatrix}\begin{pmatrix} -1 \\ -1 \end{pmatrix} =$$

$$\frac{1}{10}\begin{bmatrix} 1 \\ 3 \end{bmatrix}\begin{bmatrix} 1 & 3 \end{bmatrix}\begin{pmatrix} -1 \\ -1 \end{pmatrix} = \frac{1}{10}\begin{bmatrix} 1 & 3 \\ 3 & 9 \end{bmatrix}\begin{pmatrix} -1 \\ -1 \end{pmatrix} = \frac{1}{5}\begin{pmatrix} -2 \\ -6 \end{pmatrix};$$

```
b      = Vector([-1,-1])
B      = Matrix([[3,-1]])

a      = Homogenea(B).sgen|1
# a      = Homogenea(B).enulo.sgen|1 # alternativa equivalente
# a      = EAfin(B, V0(2)).S.sgen|1  # alternativa equivalente

A      = Matrix([a])
P      = A*InvMat((~A)*A)*(~A)      # Matriz proyección
bhat1  = P*b                        # Alternativa 1
x      = SEL( (~A)*A, (~A)*b ).solP # Solución Ecuaciones Normales
bhat2  = A*x                        # Alternativa 2
Sistema([bhat1, bhat2])
```

□

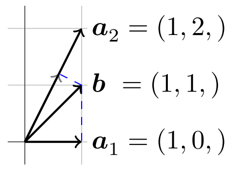
(L-12) Question 3.

$$\widehat{\boldsymbol{b}} = [\boldsymbol{a}]\left([\boldsymbol{a}]^{\intercal}[\boldsymbol{a}]\right)^{-1}[\boldsymbol{a}]^{\intercal}\boldsymbol{b} = \begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix}\left(\begin{bmatrix} -1 & 1 & -1 & 1 \end{bmatrix}\begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix}\right)^{-1}\begin{bmatrix} -1 & 1 & -1 & 1 \end{bmatrix}\begin{pmatrix} 1 \\ 2 \\ 1 \\ 3 \end{pmatrix} =$$

$$\frac{1}{4}\begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix}\begin{bmatrix} -1 & 1 & -1 & 1 \end{bmatrix}\begin{pmatrix} 1 \\ 2 \\ 1 \\ 3 \end{pmatrix} = \frac{1}{4}\begin{bmatrix} 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \end{bmatrix}\begin{pmatrix} 1 \\ 2 \\ 1 \\ 3 \end{pmatrix} = \frac{1}{4}\begin{pmatrix} -3 \\ 3 \\ -3 \\ 3 \end{pmatrix};$$

□

(L-12) Question 4(a) $\widehat{\boldsymbol{b}_1} = (1,\ \ 0,)$ and $\widehat{\boldsymbol{b}_2} = (\frac{3}{5},\ \ \frac{6}{5},)$. Then $\widehat{\boldsymbol{b}_1} + \widehat{\boldsymbol{b}_2} \neq \boldsymbol{b}$.




```
b  = Vector([1,1])
a1 = Vector([1,0])
a2 = Vector([1,2])

A1 = Matrix([a1])
bhat1 = A1 * SEL((~A1)*A1,(~A1*b)).solP

A2 = Matrix([a2])
bhat2 = A2 * SEL((~A2)*A2,(~A2*b)).solP
Sistema([bhat1, bhat2])
```

□

(L-12) Question 4(b) Since $\mathbf{A}$ is invertible, the projection matrix $\mathbf{P} = \mathbf{A}(\mathbf{A}^\intercal\mathbf{A})^{-1}\mathbf{A}^\intercal = \mathbf{I}$ projects onto all of $\mathbb{R}^2$. Therefore $\widehat{\boldsymbol{b}_1} = \mathbf{P}\boldsymbol{b}_1 = \boldsymbol{b}_1$.

```
A3 = Matrix([a1,a2])
P  = A3*InvMat((~A3)*A3)*(~A3)
bhat3 = P*b
Sistema([bhat1,bhat2,bhat3,P])
```

□

(L-12) Question 5(a) $\mathbf{P}^2 = \mathbf{P}$ and therefore $(\mathbf{I}-\mathbf{P})^2 = (\mathbf{I}-\mathbf{P})(\mathbf{I}-\mathbf{P}) = \mathbf{I}-\mathbf{P}\mathbf{I}-\mathbf{I}\mathbf{P}+\mathbf{P}^2 = \mathbf{I}-\mathbf{P}$.
When $\mathbf{P}$ projects onto the column space of $\mathbf{A}$, $(\mathbf{I}-\mathbf{P})$ projects onto the *left nullspace* of $\mathbf{A}$.

□

(L-12) Question 5(b) $\mathbf{P}^\intercal = \mathbf{P}$ and therefore $(\mathbf{I}-\mathbf{P})^\intercal = (\mathbf{I}^\intercal - \mathbf{P}^\intercal) = \mathbf{I}-\mathbf{P}$.

□

(L-12) Question 6(a)

$$\mathbf{P}_1 = \frac{1}{9}\begin{bmatrix} 1 & -2 & -2 \\ -2 & 4 & 4 \\ -2 & 4 & 4 \end{bmatrix}; \qquad \mathbf{P}_1 = \frac{1}{9}\begin{bmatrix} 4 & 4 & -2 \\ 4 & 4 & -2 \\ -2 & -2 & 1 \end{bmatrix}.$$

$\mathbf{P}_1\mathbf{P}_2 =$ zero matrix because $\boldsymbol{a}_1 \perp \boldsymbol{a}_2$.

□

(L-12) Question 6(b) $\widehat{\boldsymbol{b}_1} = \frac{1}{9}\left(1, \quad -2, \quad -2,\right)$, $\widehat{\boldsymbol{b}_2} = \frac{1}{9}\left(4, \quad 4, \quad -2,\right)$, $\widehat{\boldsymbol{b}_3} = \frac{1}{9}\left(4, \quad -2, \quad 4,\right)$. Then $\widehat{\boldsymbol{b}_1}+\widehat{\boldsymbol{b}_2}+\widehat{\boldsymbol{b}_3} = \left(1, \quad 0, \quad 0,\right) = \boldsymbol{b}$. Note that $\boldsymbol{a}_3 \perp \boldsymbol{a}_1$ and $\boldsymbol{a}_3 \perp \boldsymbol{a}_2$.

□

(L-12) Question 6(c)

$$\mathbf{P}_1 + \mathbf{P}_2 + \mathbf{P}_3 = \frac{1}{9}\begin{bmatrix} 1 & -2 & -2 \\ -2 & 4 & 4 \\ -2 & 4 & 4 \end{bmatrix} + \frac{1}{9}\begin{bmatrix} 4 & 4 & -2 \\ 4 & 4 & -2 \\ -2 & -2 & 1 \end{bmatrix} + \frac{1}{9}\begin{bmatrix} 4 & -2 & 4 \\ -2 & 1 & -2 \\ 4 & -2 & 4 \end{bmatrix} = \mathbf{I}.$$

□

(L-12) Question 7(a) $\widehat{\boldsymbol{b}_1} = \mathbf{A}_1\left(\mathbf{A}_1^\intercal\mathbf{A}_1\right)^{-1}(\mathbf{A}_1^\intercal)\boldsymbol{b}_1 = \left(2, \quad 3, \quad 0,\right)$ and $\widehat{[1]}e = \left(0, \quad 0, \quad 4,\right)$.

□

(L-12) Question 7(b) $\widehat{\boldsymbol{b}_2} = \mathbf{A}_2\left(\mathbf{A}_2^\intercal\mathbf{A}_2\right)^{-1}(\mathbf{A}_2^\intercal)\boldsymbol{b}_2 = \left(4, \quad 4, \quad 6,\right)$ and $\widehat{\boldsymbol{e}_2} = \left(0, \quad 0, \quad 0,\right)$.

□

(L-12) Question 7(c)

$$\mathbf{P}_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ projection on } xy \text{ plane.} \qquad \mathbf{P}_2 = \frac{1}{2}\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} = \left(\mathbf{P}_2\right)^2.$$

□